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Point Estimation of Root Finding Methods

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To Ljiljana

Preface

The problem of solving nonlinear equations and systems of equations ranks among the most significant in the theory and practice, not only of applied mathematics but also of many branches of engineering sciences, physics, computer science, astronomy, finance, and so on. A glance at the bibliography and the list of great mathematicians who have worked on this topic points to a high level of contemporary interest. Although the rapid development of digital computers led to the effective implementation of many numerical methods, in practical realization, it is necessary to solve various problems such as computational efficiency based on the total central processor unit time, the construction of iterative methods which possess a fast convergence in the presence of multiplicity (or clusters) of a desired solution, the control of rounding errors, information about error bounds of obtained approximate solution, stating computationally verifiable initial conditions that ensure a safe convergence, etc. It is the solution of these challenging problems that was the principal motivation for the present study.

In this book, we are mainly concerned with the statement and study of initial conditions that provide the guaranteed convergence of an iterative method for solving equations of the form $f(z) = 0$. The traditional approach to this problem is mainly based on asymptotic convergence analysis using some strong hypotheses on differentiability and derivative bounds in a rather wide domain. This kind of conditions often involves some unknown parameters as constants, or even desired roots of equation in the estimation procedure. Such results are most frequently of theoretical importance and they provide only a qualitative description of the convergence property. The first results dealing with the computationally verifiable domain of convergence were obtained by Smale (1981), Smale (1986), Shub and Smale (1985), and Kim (1985). This approach, often referred to as "point estimation theory," treats convergence conditions and the domain of convergence in solving an equation $f(z) = 0$ using only the information of f at the initial point $\boldsymbol{z}^{(0)}$.

In 1981, Smale introduced the concept of an *approximate zero* as an initial point which provides the safe convergence of Newton's method. Later, in 1986,

he considered the convergence of Newton's method from data at a single point. X. Wang and Han (1989) and D. Wang and Zhao (1995) obtained some improved results. The study in this field was extended by Kim (1988) and Curry (1989) to some higher-order iterative methods including Euler's method and Halley's method, and by Chen (1989), who dealt with the general Newton-like quadratically convergent iterative algorithms. A short review of these results is given in the first part of Chap. 2. Wang–Zhao's improvement of Smale's convergence theorem and an interesting application to the Durand–Kerner method for the simultaneous determination of polynomial zeros are presented in the second part of Chap. 2.

The main aim of this book is to state such quantitative initial conditions for predicting the immediate appearance of the guaranteed and fast convergence of the considered numerical algorithm. Special attention is paid to the convergence analysis of iterative methods for the simultaneous determination of the zeros of algebraic polynomials. However, the problem of the choice of initial approximations which ensure a safe convergence is a very difficult one and it cannot be solved in a satisfactory way in general, not even in the case of simple functions, such as algebraic polynomials. In 1995, the author of this book and his contributors developed two procedures to state initial conditions for the safe convergence of simultaneous methods for finding polynomial zeros. The results were based on suitable localization theorems for polynomial zeros and the convergence of error sequences. Chapter 3 is devoted to initial conditions for the guaranteed convergence of most frequently used iterative methods for the simultaneous approximations of all simple zeros of algebraic polynomials. These conditions depend only on the coefficients of a given polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ of degree n and the vector of initial approximations $\boldsymbol{z}^{(0)} = \big(z_1^{(0)}, \ldots, z_n^{(0)}\big)$. In particular, some efficient a posteriori error bound methods that produce disks containing the sought zeros and require fewer numerical operations than the corresponding ordinary interval methods are considered in the last part of Chap. 3.

The new results presented in Chaps. 4 and 5 are concerned with the higher-order families of methods for the simultaneous determination of complex zeros. These methods are based on the iterative formula of Hansen–Patrick's type for finding a single zero. As in Chap. 3, we state computationally verifiable initial conditions that guarantee the convergence of the presented methods. Initial conditions ensuring convergence of the corresponding iterative methods for the inclusion of polynomial zeros are established in Chap. 5. Convergence behavior of the considered methods is illustrated by numerical examples.

I wish to thank Professor C. Carstensen of Humboldt University in Berlin. Our joint work (*Numer. Math.* 1995) had a stimulating impact on the development of the basic ideas for obtaining some results given in this book. I am grateful to Professor S. Smale, the founder of the point estimation theory, who drew my attention to his pioneering work. I am also thankful to my contributors and coauthors of joint papers Professor T. Sakurai of the

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