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# Mathematical Theory of Feynman Path Integrals

## An Introduction

2nd corrected and enlarged edition

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## Preface to the Second Edition

This second edition, unfortunately, had to be done without the contribution of Raphael Høegh-Krohn, who died on 28 January 1988. The authors of the present edition hope very much that the result of their efforts would had been appreciated by him. His beloved memory has been a steady inspiration to us. Since the appearance of the first edition many new developments have taken place. The present edition tries to take this into account in several ways, keeping however the basic structure and contents of the first edition. At that time the book was the first rigorous one to appear in the area and was written in a sort of pioneering spirit. In our opinion it is still valid as an introduction to all the work which followed; therefore in this second edition we preserve its form entirely (except for correcting some misprints and slightly improving some formulations). A chapter has been however added, in which many new developments are included. These concern both new mathematical developments in the definition and properties of the integrals, and new exciting applications to areas like low dimensional topology and quantized gauge fields. In addition we have added historical notes to each of the chapters and corrected several misprints of the previous edition. As for references, we have kept all those of the first edition, numbered from 1 to 56 (with the corresponding updating), and added new references (in alphabetic order).

We are very grateful to many coworkers, friends and colleagues, who inspired us in a number of ways. Special thanks are due to Philippe Blanchard, Zdzisław Brzeźniak, Luca Di Persio, Jorge Rezende, Jörg Schäfer, Ambar Sengupta, Ludwig Streit, Aubrey Truman, Luciano Tubaro, and Jean-Claude Zambrini. We also like to remember with gratitude the late Yuri L. Daleckii and Michel Sirugue who gave important contributions to this area of research.

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# Preface to the First Edition

In this work we develop a general theory of oscillatory integrals on real Hilbert spaces and apply it to the mathematical foundation of the so-called Feynman path integrals of non-relativistic quantum mechanics, quantum statistical mechanics and quantum field theory. The translation invariant integrals we define provide a natural extension of the theory of finite dimensional oscillatory integrals, which has recently undergone an impressive development, and appear to be a suitable tool in infinite dimensional analysis. For example, on the basis of the present work, we have extended the methods of stationary phase, Lagrange immersions and corresponding asymptotic expansions to the infinite dimensional case, covering in particular the expansions around the classical limit of quantum mechanics. A particular case of the oscillatory integrals studied in the present work are the Feynman path integrals used extensively in physics literature, starting with the basic work on quantum dynamics by Dirac and Feynman, in the 1940s.

In the introduction, we give a brief historical sketch and some references concerning previous work on the problem of the mathematical justification of Feynman's heuristic formulation of the integral. However, our aim with the present publication was not to write a review work, but rather to develop from scratch a self-contained theory of oscillatory integrals in infinite dimensional spaces, in view of the mathematical and physical applications mentioned above.

The structure of the work is briefly as follows. It consists of nine chapters. Chapter 1 is the introduction. Chapters 2 and 4 give the definitions and basic properties of the oscillatory integrals, which we call Fresnel integrals or normalized integrals, for the cases where the phase function is a bounded perturbation of a non-degenerate quadratic form (positive in Chap. 2). Chapters 3 and 5–9 give applications to quantum mechanics, namely $N$-particle systems with bounded potentials (Chap. 3) and systems of harmonic oscillators with finitely or infinitely many degrees of freedom (Chaps. 5–9), with relativistic quantum fields as a particular case (Chap. 9).

This work appeared first as a Preprint of the Mathematics Institute of Oslo University, in October 1974.

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