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Representation Theory and Complex Analysis

Lectures given at the
C.I.M.E. Summer School
held in Venice, Italy
June 10–17, 2004

Editors: Enrico Casadio Tarabusi
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Preface

This volume collects the notes of six series of lectures given on the occasion of the CIME session *Representation Theory and Complex Analysis* held in Venice on July 10–17, 2004. We thank Venice International University for its hospitality at the beautiful venue of San Servolo island.

Our aim in organizing this meeting was to present the audience with a wide spectrum of recent results on the subject of the title, ranging from topics with an analytical flavor, to more algebraic or geometric oriented ones, without neglecting interactions with other domains, such as quantum computing.

Two papers present a general introduction to ideas and properties of analysis on semi-simple Lie groups and their unitary representations. MICHAEL COWLING presents a panorama of various interactions between representation theory and harmonic analysis on semisimple groups and symmetric spaces. Unexpected phenomena occur in this context, as for instance the Kunze–Stein property, that reveal a dramatic difference between these groups and group actions and the classical amenable group (an extension of abelian groups). Results of this type are strongly related to the vanishing of coefficients of unitary representations. Complementarily, ALAIN VALETTE recalls the notion of amenability and investigates its relations with vanishing of coefficients of unitary representations of semi-simple groups and with ergodic actions. He applies these ideas to show another surprising property of representations of semi-simple groups and their lattices, namely Margulis’ super-rigidity.

Three papers deal in full detail with the hard analysis of semisimple group representations. Ideally, this analysis could be split into representations of real groups or complex groups, or of algebraic groups over local fields. A deep account of the interaction between the real and complex world is given by MASAKI KASHIWARA, whose paper studies the relations between the representation theory of real semisimple Lie groups and the (microlocal) geometry of the flag manifolds associated with the corresponding complex algebraic groups. These results, a considerable part of which are joint work with W. Schmid, were announced some years ago, and are published here in

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complete form for the first time. DAVID VOGAN expresses unitary representations of real or complex semi-simple groups using tools of complex analysis, such as minimal globalizations realized on Dolbeault cohomology with compact supports. EDWARD FRENKEL describes the geometric Langlands correspondence for complex algebraic curves, concentrating on the ramified case where a finite number of regular singular points is allowed.

Finally, NOLAN WALLACH illustrates briefly a surprising application that could be relevant for the future of computing and its complexity: his paper studies how representation theory is related to quantum computing, focusing attention in particular on the study of qubit entanglement.

We wish to thank all the lecturers for the excellence of their live and written contributions, as well as the many participants from all age ranges and parts of the world, who created a very pleasant working atmosphere.

Roma and Venezia, November 2006

*Enrico Casadio Tarabusi
Andrea D'Agnolo
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