Lecture Notes in Mathematics 1927

C.I.M.E. means Centro Internazionale Matematico Estivo, that is, International Mathematical Summer Center. Conceived in the early fifties, it was born in 1954 and made welcome by the world mathematical community where it remains in good health and spirit. Many mathematicians from all over the world have been involved in a way or another in C.I.M.E.'s activities during the past years.

So they already know what the C.I.M.E. is all about. For the benefit of future potential users and co-operators the main purposes and the functioning of the Centre may be summarized as follows: every year, during the summer, Sessions (three or four as a rule) on different themes from pure and applied mathematics are offered by application to mathematicians from all countries. Each session is generally based on three or four main courses ($24-30$ hours over a period of 6-8 working days) held from specialists of international renown, plus a certain number of seminars.

A C.I.M.E. Session, therefore, is neither a Symposium, nor just a School, but maybe a blend of both. The aim is that of bringing to the attention of younger researchers the origins, later developments, and perspectives of some branch of live mathematics.

The topics of the courses are generally of international resonance and the participation of the courses cover the expertise of different countries and continents. Such combination, gave an excellent opportunity to young participants to be acquainted with the most advance research in the topics of the courses and the possibility of an interchange with the world famous specialists. The full immersion atmosphere of the courses and the daily exchange among participants are a first building brick in the edifice of international collaboration in mathematical research.

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Nicola Fusco

Calculus of Variations and Nonlinear Partial Differential Equations

Lectures given at the
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With a historical overview by Elvira Mascolo

Editors: Bernard Dacorogna, Paolo Marcellini

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Preface

We organized this CIME Course with the aim to bring together a group of top leaders on the fields of *calculus of variations* and *nonlinear partial differential equations*. The list of speakers and the titles of lectures have been the following:

- Luigi Ambrosio, *Transport equation and Cauchy problem for non-smooth vector fields.*
- Luis A. Caffarelli, *Homogenization methods for non divergence equations.*
- Michael Crandall, *The infinity-Laplace equation and elements of the calculus of variations in L-infinity.*
- Gianni Dal Maso, *Rate-independent evolution problems in elasto-plasticity: a variational approach.*
- Lawrence C. Evans, *Weak KAM theory and partial differential equations.*
- Nicola Fusco, *Geometrical aspects of symmetrization.*

In the original list of invited speakers the name of Pierre Louis Lions was also included, but he, at the very last moment, could not participate.

The Course, just looking at the number of participants (more than 140, one of the largest in the history of the CIME courses), was a great success; most of them were young researchers, some others were well known mathematicians, experts in the field. The high level of the Course is clearly proved by the quality of notes that the speakers presented for this Springer Lecture Notes.

We also invited Elvira Mascolo, the CIME scientific secretary, to write in the present book an overview of the history of CIME (which she presented at Cetraro) with special emphasis in calculus of variations and partial differential equations.

Most of the speakers are among the world leaders in the field of *viscosity solutions* of partial differential equations, in particular nonlinear pde's of *implicit type*. Our choice has not been random; in fact we and other mathematicians have recently pointed out a theory of *almost everywhere solutions* of pde's of *implicit type*, which is an approach to solve *nonlinear systems* of pde's. Thus this Course has been an opportunity to bring together experts of viscosity solutions and to see some recent developments in the field.

We briefly describe here the articles presented in this Lecture Notes.

Starting from the lecture by Luigi Ambrosio, where the author studies the well-posedness of the Cauchy problem for the homogeneous conservative continuity equation

$$\frac{d}{dt}\mu_t + D_x \cdot (b\mu_t) = 0\,, \qquad (t,x) \in I \times \mathbb{R}^d$$

and for the transport equation

$$\frac{d}{dt}w_t + b \cdot \nabla w_t = c_t\,,$$

where $b(t,x) = b_t(x)$ is a given time-dependent vector field in $\mathbb{R}^d$. The interesting case is when $b_t(\cdot)$ is not necessarily Lipschitz and has, for instance, a Sobolev or BV regularity. Vector fields with this "low" regularity show up, for instance, in several PDE's describing the motion of fluids, and in the theory of conservation laws.

The lecture of Luis Caffarelli gave rise to a joint paper with Luis Silvestre; we quote from their introduction:

"When we look at a differential equation in a very irregular media (composite material, mixed solutions, etc.) from very close, we may see a very complicated problem. However, if we look from far away we may not see the details and the problem may look simpler. The study of this effect in partial differential equations is known as *homogenization*. The effect of the inhomogeneities oscillating at small scales is often not a simple average and may be hard to predict: a geodesic in an irregular medium will try to avoid the bad areas, the roughness of a surface may affect in nontrivial way the shapes of drops laying on it, etc... The purpose of these notes is to discuss three problems in homogenization and their interplay.

In the first problem, we consider the homogenization of a free boundary problem. We study the shape of a drop lying on a rough surface. We discuss in what case the homogenization limit converges to a perfectly round drop. It is taken mostly from the joint work with Antoine Mellet *(see the precise references in the article by Caffarelli and Silvestre in this lecture notes)*. The second problem concerns the construction of plane like solutions to the minimal surface equation in periodic media. This is related to homogenization of minimal surfaces. The details can be found in the joint paper with Rafael de la Llave. The third problem concerns existence of homogenization limits for solutions to fully nonlinear equations in ergodic random media. It is mainly based on the joint paper with Panagiotis Souganidis and Lihe Wang.

We will try to point out the main techniques and the common aspects. The focus has been set to the basic ideas. The main purpose is to make this advanced topics as readable as possible."

Michael Crandall presents in his lecture an outline of the theory of the archetypal L^∞ variational problem in the calculus of variations. Namely, given

an open $U \subset \mathbb{R}^n$ and $b \in C(\partial U)$, find $u \in C(\overline{U})$ which agrees with the boundary function b on ∂U and minimizes

$$\mathcal{F}_\infty(u,U) := \| |Du| \|_{L^\infty(U)}$$

among all such functions. Here $|Du|$ is the Euclidean length of the gradient Du of u. He is also interested in the "Lipschitz constant" functional as well: if K is any subset of $\mathbb{R}^n$ and $u : K \to \mathbb{R}$, its least Lipschitz constant is denoted by

$$\mathrm{Lip}\,(u,K) := \inf \{ L \in \mathbb{R} : \ |u(x) - u(y)| \leq L\,|x-y|\,,\ \forall x,y \in K \}\,.$$

One has $\mathcal{F}_\infty(u,U) = \mathrm{Lip}\,(u,U)$ if U is *convex*, but equality does not hold in general.

The author shows that a function which is absolutely minimizing for Lip is also absolutely minimizing for $\mathcal{F}_\infty$ and conversely. It turns out that the absolutely minimizing functions for Lip and $\mathcal{F}_\infty$ are precisely the viscosity solutions of the famous partial differential equation

$$\Delta_\infty u \ = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0\,.$$

The operator Δ_∞ is called the "∞-Laplacian" and *"viscosity solutions"* of the above equation are said to be $\infty-$*harmonic*.

In his lecture Lawrence C. Evans introduces some new PDE methods developed over the past 6 years in so-called *"weak KAM theory"*, a subject pioneered by J. Mather and A. Fathi. Succinctly put, the goal of this subject is the employing of dynamical systems, variational and PDE methods to find "integrable structures" within general Hamiltonian dynamics. Main references *(see the precise references in the article by Evans in this lecture notes)* are Fathi's forthcoming book and an article by Evans and Gomes.

Nicola Fusco in his lecture presented in this book considers two model functionals: the *perimeter* of a set E in $\mathbb{R}^n$ and the *Dirichlet integral* of a scalar function u. It is well known that on replacing E or u by its *Steiner symmetral* or its *spherical symmetrization*, respectively, both these quantities decrease. This fact is classical when E is a smooth open set and u is a C^1 function. On approximating a set of finite perimeter with smooth open sets or a Sobolev function by C^1 functions, these inequalities can be extended by lower semicontinuity to the general setting. However, an approximation argument gives no information about the equality case. Thus, if one is interested in understanding when equality occurs, one has to carry on a deeper analysis, based on fine properties of sets of finite perimeter and Sobolev functions. Briefly, this is the subject of Fusco's lecture.

Finally, as an appendix to this CIME Lecture Notes, as we said Elvira Mascolo, the CIME scientific secretary, wrote an interesting overview of the history of CIME having in mind in particular *calculus of variations* and PDES.

We are pleased to express our appreciation to the speakers for their excellent lectures and to the participants for contributing to the success of the Summer School. We had at Cetraro an interesting, rich, nice, friendly atmosphere, created by the speakers, the participants and by the CIME organizers; also for this reason we like to thank the Scientific Committee of CIME, and in particular Pietro Zecca (CIME Director) and Elvira Mascolo (CIME Secretary). We also thank Carla Dionisi, Irene Benedetti and Francesco Mugelli, who took care of the day to day organization with great efficiency.

Bernard Dacorogna *and* Paolo Marcellini

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