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Stability of Nonautonomous Differential Equations

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To our parents

Preface

The main theme of this book is the stability of nonautonomous differential equations, with emphasis on the study of the existence and smoothness of invariant manifolds, and the Lyapunov stability of solutions. We always consider a nonuniform exponential behavior of the linear variational equations, given by the existence of a nonuniform exponential contraction or a nonuniform exponential dichotomy. Thus, the results hold for a much larger class of systems than in the "classical" theory of exponential dichotomies.

The departure point of the book is our joint work on the construction of invariant manifolds for nonuniformly hyperbolic trajectories of nonautonomous differential equations in Banach spaces. We then consider several related developments, concerning the existence and regularity of topological conjugacies, the construction of center manifolds, the study of reversible and equivariant equations, and so on. The presentation is self-contained and intends to convey the full extent of our approach as well as its unified character. The book contributes towards a rigorous mathematical foundation for the theory in the infinite-dimensional setting, also with the hope that it may lead to further developments in the field. The exposition is directed to researchers as well as graduate students interested in differential equations and dynamical systems, particularly in stability theory.

The first part of the book serves as an introduction to the other parts. After giving in Chapter 1 a detailed introduction to the main ideas and motivations behind the theory developed in the book, together with an overview of its contents, we introduce in Chapter 2 the concept of nonuniform exponential dichotomy, which is central in our approach, and we discuss some of its basic properties. Chapter 3 considers the problem of the robustness of nonuniform exponential dichotomies.

In the second part of the book we discuss several consequences of local nature for a nonlinear system when the associated linear variational equation admits a nonuniform exponential dichotomy. In particular, we establish in Chapter 4 the existence of Lipschitz stable manifolds for nonautonomous equations in a Banach space. In Chapters 5 and 6 we establish the smooth-

ness of the stable manifolds. We first consider the finite-dimensional case in Chapter 5, with the method of invariant families of cones. This approach uses in a decisive manner the compactness of the closed unit ball in the ambient space, and this is why we consider only finite-dimensional spaces in this chapter. Moreover, the proof strongly relies on the use of Lyapunov norms to control the nonuniformity of the exponential dichotomies. As an outcome of our approach we provide examples of C^1 vector fields with invariant stable manifolds, while in the existing nonuniform hyperbolicity theory one assumes that the vector field is of class $C^{1+\alpha}$. In Chapter 6 we consider differential equations in Banach spaces, although at the expense of slightly stronger assumptions for the vector field. The method of proof is different from the one in Chapter 5, and is based on the application of a lemma of Henry to obtain both the existence and smoothness of the stable manifolds using a single fixed point problem. In addition, we show that not only the trajectories but also their derivatives with respect to the initial condition decay with exponential speed along the stable manifolds. A feature of our approach is that we deal directly with flows or semiflows instead of considering the associated time-1 maps. In Chapter 7 we establish a version of the Grobman–Hartman theorem for nonautonomous differential equations in Banach spaces, assuming that the linear variational equation admits a nonuniform exponential dichotomy. In addition, we show that the conjugacies that we construct are always Hölder continuous.

The third part of the book is dedicated to the study of center manifolds. In Chapter 8 we extend the approach in Chapter 6 to nonuniform exponential trichotomies, and we establish the existence of center manifolds that are as smooth as the vector field. In particular, we obtain simultaneously the existence and smoothness of the center manifolds using a single fixed point problem. In Chapter 9 we show that some symmetries of the differential equations descend to the center manifolds. More precisely, we consider the properties of reversibility and equivariance in time, and we show that the dynamics on the center manifold is reversible or equivariant if the dynamics in the ambient space has the same property.

In the fourth part of the book we study the so-called regularity theory of Lyapunov and its applications to the stability theory of differential equations. We note that this approach is distinct from what is usually called Lyapunov's second method, which is based on the use of Lyapunov functions. In Chapter 10 we provide a detailed exposition of the regularity theory, organized in a pragmatic manner so that it can be used in the last two chapters of the book. In Chapter 11 we extend the regularity theory to the infinite-dimensional setting of Hilbert spaces. Chapter 12 is dedicated to the study of the stability of nonautonomous differential equations using the regularity theory. We note that the notion of Lyapunov regularity is much less restrictive than the notion of uniform stability, and thus we obtain the persistence of the stability of solutions of nonautonomous differential equations under much weaker assumptions.

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