Lecture Notes in Mathematics 1914

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Affine Density in Wavelet Analysis

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Dedicated to

my Parents

Preface

During the last 20 years, wavelet analysis has become a major research area in mathematics, not only because of the beauty of the mathematical theory of wavelet systems (sometimes also called affine systems), but also because of its significant impact on applications, especially in signal and image processing. After the extensive exploration of orthonormal bases of classical affine systems that has occupied much of the history of wavelet theory, recently both *wavelet frames* — redundant wavelet systems — and *irregular wavelet systems* — wavelet systems with an arbitrary sequence of time-scale indices — have come into focus as a main area of research. Two main reasons for this are to serve new applications which require robustness against noise and erasures, and to derive a deeper understanding of the theory of classical affine systems. However, a comprehensive theory to treat irregular wavelet frames does not exist so far. The main difficulty consists of the highly sensitive interplay between geometric properties of the sequence of time-scale indices and frame properties of the associated wavelet system.

In this research monograph, we introduce the new notion of affine density for sequences of time-scale indices to wavelet analysis as a highly effective tool for studying irregular wavelet frames. We present many results concerning the structure of weighted irregular wavelet systems with finitely many generators, adding considerably to our understanding of the relation between the geometry of the time-scale indices of these general wavelet systems and their frame properties.

This book is the author's Habilitationsschrift in mathematics at the Justus-Liebig-Universität Gießen. It is organized as follows. The introduction presents a detailed overview of the recent developments in the study of irregular wavelet frames and of the already quite established theory of the relation between Beurling density and the geometry of sequences of time-frequency indices of Gabor systems. Furthermore, it explains our main results in an informal way. Chapter 2 reviews the terminology and notations from

frame theory as well as from wavelet and time-frequency analysis employed in this book.

The notion of weighted affine density, which will turn out to be a most effective tool for studying the geometry of sequences of time-scale indices associated with weighted irregular wavelet systems, will be introduced in Chapter 3. We illustrate the new notion by giving several examples. We further compare this notion of affine density with the affine density that was independently and simultaneously introduced by Sun and Zhou [119] and point out the advantages of our notion.

In Chapter 4, we prove that the notion of weighted affine density leads to very elegant necessary conditions for the existence of general wavelet frames on the sequence of time-scale indices. The usefulness of this notion is emphasized by its utility for the study of a rather technical-appearing hypothesis known as the *local integrability condition (LIC)* of a characterization result for weighted wavelet Parseval frames by Hernández, Labate, and Weiss [77]. In fact, we show that under a mild regularity assumption on the analyzing wavelets, the LIC is in fact solely a density condition.

Chapter 5 is devoted to the study of a quantitative relation between frame bounds and affine density conditions, since the complexity of frame algorithms is strongly related to the values of the frame bounds. A striking result here is a fundamental relationship between the affine density of the sequence of time-scale indices, the frame bounds, and the admissibility constant of a weighted irregular wavelet frame with finitely many generators. Several implications of this result are outlined, among which is the revelation of a reason for the non-existence of a Nyquist phenomenon for wavelet systems and the uniformity of sequences of time-scale indices associated with tight wavelet frames. In addition, we also present the first result in which the existence of particular wavelet frames is completely characterized by density conditions. The non-existence of very general co-affine frames is then shown to follow as a corollary.

In Chapter 6, we show that most irregular wavelet frames (and even wavelet Schauder bases) satisfy a so-called *Homogeneous Approximation Property (HAP)*. This property not only implies certain invariance properties under time-scale shifts when approximating with wavelet frames, but is also shown to have impact on density considerations. In addition to these main results, our techniques introduce some very useful new tools for the study of wavelet systems, e.g., certain Wiener amalgam spaces and — related with these objects — a particular class of analyzing wavelets.

Chapter 7 is devoted to the study of shift-invariance, i.e., invariance under integer translations, which is a desirable feature for many applications, since this ensures that similar structures in a signal are more easily detectable. The oversampling theorems from wavelet analysis show that most classical affine systems can be turned into a shift-invariant wavelet system with comparable frame properties. Most interestingly, the process also leaves density properties invariant, and the question concerning necessity of this fact for irregular wavelet systems arises. In this chapter we study the analog of this problem in

time-frequency analysis and give a complete answer for irregular Gabor systems. Along the way we introduce a new notion of weighted Beurling density and derive extensions of results from H. Landau [97], and Balan, Casazza, Heil, and Z. Landau [7]. The results obtained in this chapter are not only interesting by itself, but can also be regarded as an important step towards the study of similar questions in wavelet analysis.

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