Part III. Optimal Reduction and Singular Reduction by Stages

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In this part we will apply the techniques recently introduced by see Ortega and Ratiu [2002] and Ortega [2003a], which are based on the theory of distributions to generalize the results presented in Part II as well as other various symplectic reduction methods due to Marsden, Weinstein, Sjamaar, Bates, Lerman, Marle, Kazhdan, Kostant, and Sternberg. This approach is based on the definition of an object, that will be called the *optimal momentum map*, that to some extent generalizes the standard momentum map that we used previously in this book. One of the advantages of this point of view is its degree of generality which will allow us to construct symplectic point and orbit reduced spaces purely within the Poisson category under hypotheses that do not necessarily imply the existence of a standard or group valued (Alekseev, Malkin, and Meinrenken [1998]) momentum map. All along this part we will refer to the construction of symplectic reduced spaces with the help of the optimal momentum map as ***optimal reduction***.

This part is divided into three chapters. The first chapter contains a brief description of the optimal momentum map and explains how to construct symplectic or Marsden–Weinstein reduced spaces in the context in which this object is defined. Most of the definitions and results in this preliminary chapter are contained in Ortega and Ratiu [2002]; Ortega [2002].

The second chapter of this part explores in the context of the optimal momentum map the orbit reduction procedure. Orbit reduction is an approach to symplectic reduction equivalent to the Marsden–Weinstein

reduction that we used in Part II. It consists of considering in the reduction process the inverse images by the momentum map of coadjoint orbits instead of just momentum values; this set is, under certain hypotheses, a smooth G–invariant space whose corresponding orbit space is symplectomorphic to the Marsden–Weinstein reduced space. In order to make a distinction between these two reduction procedures we will talk about ***point*** and ***orbit reduction***. The expression for the symplectic structure of the orbit reduced space puts into relation the symplectic form of the original manifold with the so called natural Kostant–Kirillov–Souriau symplectic form of the coadjoint orbit that we used to construct it. Our goal in this chapter will be the reproduction of this scheme in the context of the optimal momentum map. This will need the introduction of smooth (pre)–symplectic manifolds that will generalize to this context the coadjoint orbits and their natural symplectic structures. We will refer to these as ***polar reduced spaces*** due to their close ties with the notion of polarity introduced by the author in his study of singular dual pairs (Ortega [2003a]). In particular, we will see that there is an interesting interplay between the so called von Neumann condition for a canonical group action and the polar reduction scheme.

The last chapter of this Part III will extend to the optimal context the reduction by stages procedure that we studied in Part II for the standard momentum map. The advantages in terms of generality presented by the optimal momentum map will allow us to formulate a reduction by stages theorem for any normal subgroup of a Lie group acting canonically on a Poisson manifold for which, in principle, there is no associated standard momentum map. At the end of this chapter, an motivated by its importance in terms of applications we will study the particular case of a Hamiltonian proper action on a symplectic manifold for which we do not impose freeness. These results provide the generalization to the singular context of the reduction by stages theorems in Part II.