

# Lecture Notes in Mathematics

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# Iterative Approximation of Fixed Points

 Springer

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To my parents

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## Preface to the Second Edition

This is a revised and enlarged version of the second printing (with up-dated bibliography, 2004) of the first edition, published by *Efemeride* in 2002.

All chapters of the book were practically revised in a certain extent and some new Sections were also added with the aim to improve the coverage of the topic and to attain its main aim: to summarize in a gradual and natural way the most significant contributions to the approximation of fixed points of nonlinear contractive type mappings, by presenting, for each important iterative method, some of the most relevant, interesting and actual results.

Only *constructive* fixed point theorems are mainly the subject of the book. A constructive fixed point theorem establishes not only the existence (and possibly, uniqueness) of the fixed points, but also provides a method for approximating these fixed points and, moreover, offers information on the data dependence of the fixed points (or, alternatively, on the stability of the fixed point iterative methods).

## Main Changes in the Second Edition

1. Since the first edition had no exercises explicitly formulated, we selected and included in the new edition a number of 111 Exercises, Applications and Miscellaneous Results, distributed to all chapters, which completes the topic treated in each chapter or indicate other related directions of research.

2. A number of 7 new sections were added (3.5, 5.5, 6.4, 6.5, 9.3, 9.4, 9.5) or enlarged; section 4.4 merged section 4.2 to form a new section 4.2. Practically, all sections were significantly revised. Section 6.3 changed the title from “Ergodic fixed point iteration procedures” to “Ergodic and other fixed point iteration procedures”; section 7.3 changed the name from “Continuous dependence of the fixed points” to “Data dependence of fixed points”; Chapter 8 changed the title from “Applications of some fixed point iteration procedures” to “Iterative solution of nonlinear operator equations”, to indicate more clearer the area of applications.

3. We also simplified several proofs and corrected many of the typos.
4. We enlarged and improved Chapter 9 with some very recent new results related to the numerical comparison of fixed point iteration procedures.
5. We added other numerical examples in Chapter 9, obtained by means of the software package FIXPOINT.
6. We inserted new information in the Bibliographical Comments sections in Chapters 3-9.
7. We up-dated significantly the bibliography: more than 500 new entries were added; at the same time, some of the old entries in the first edition, now considered to be not directly related to the main topic, were eliminated. In comparison to the first edition, which had about 1050 references at the bibliography, in the present edition it considerably increased: it contains now more than 1575 titles. The bibliography itself could show how dynamic this field of research is: 1481 titles, representing 94% of the whole bibliography, were published in the last 35 years (1970-2005); 1294 of the latter, representing 82% of the whole bibliography, were published in the last 25 years (1980-2005); 1059 of them, representing 67% of the whole bibliography, were published in the last 15 years (1990-2005), while 876 titles, that is, almost 50% of the total bibliography, were published in the last 10 years (1995-2005).

The decade 1990-1999 has doubled the bibliography of the previous one (1980-1989), while the last half decade 2000-2004 produced much more than the whole decade 1990-1999.

Note that, the very recent publications (on 2005, 2006 and 2007) are partially covered in the present list of references, with only 54 titles.

## Main Merits of the Present Edition

The main merits of the current edition consist not only in a better presentation of the material, but especially in the fact that we tried to introduce and systematically apply some firm criteria of evaluating, judging and presenting the vast material existing in literature.

This enabled us, in Sections 5.5, 6.4, 6.5, 9.3, 9.4 and 9.5, to indicate some new directions of investigation of real and significant interest in the subject, and also to mention those topics which, in our opinion, are less important for theoretical and numerical purposes.

Chapter 9, devoted to error analysis of iterative methods, as well as sections 3.5, 5.5, 6.4, 6.5, include very recent, new and important results that could put into a new light the future research in the area.

In order to give an overview of the huge research work, see the data above, emphasis is put mainly on the generic results regarding the main topic, but the author's intention was to produce an as in-depth and up-to-date coverage as possible of the most significant 400 recent articles in that area.

From a huge amount of bibliography - more than 1575 entries are included in the present edition, as mentioned before - in principle only innovative research was selected and presented in the book.

## More Acknowledgments

I want to thank again Professor Ioan A. Rus, this time for carefully reading the first edition and making numerous and valuable remarks and suggestions for improving the book. I thank also Dr. Sorin Iuliu Pop for the help given at the completion of the bibliography.

Thanks are due to my PhD students Ioana Banc, Marina Bic, Natalia Jurja and Monica Laurant for reading the manuscript carefully and providing a list of typos which have now been corrected.

Baia Mare

*Vasile BERINDE*  
December 22, 2006

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## Preface to the First Edition

The literature of the last four decades abounds with papers which establish fixed point theorems for selfmaps or nonselfmaps satisfying a variety of contractive type conditions on several ambient spaces.

Having in view that many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator, on the one hand, and the fact that contractive (Lipschitzian) type conditions naturally arise for many of these problems, on the other hand, the metrical fixed point theory has developed significantly in the second part of the XX<sup>th</sup> century.

A plethora of *metrical* fixed point theorems have been obtained, more or less important from a theoretical point of view, which establish usually the existence, or the existence and uniqueness of fixed points for a certain contractive operator. Among these fixed point theorems, only a small number are important from a practical point of view, that is, they offer a *constructive* method for finding the fixed points. Among the last ones only a few give information on the error estimate (the rate of convergence) of the method.

However, from a practical point of view it is important not only to know the fixed point exists (and, possibly, is unique), but also to be able to construct that fixed point(s). As the constructive methods used in metrical fixed point theory are prevailingly *iterative* procedures, that is, *approximate* methods, it is also of crucial importance to have a priori or / and a posteriori error estimates (or, alternatively, rate of convergence) for such a method.

Starting from these numerical commands, the book aims to survey some of the most used fixed point iteration procedures: the Picard iteration, the Krasnoselskij iteration, the Mann iteration, the Ishikawa iteration etc.

The present version of the book arose out of a rather long personal research experience as well as of a Master degree course “Methods for approximating fixed points” and of a graduate course entitled “Fixed point theory”.

The last one was taught by the author to students in the Mathematics programmes at the North University Baia Mare, since 1996.



In author's opinion, the monograph is undoubtedly a *provisional* introductory approach to iterative approximation of fixed points.

With a view to its next improved and revised version(s), we shall welcome any comments, remarks, suggestions and additional bibliographical references coming with criticism from the readers.

## Acknowledgments

I am deeply indebted to Professor Ioan A. Rus from "Babes-Bolyai" University in Cluj-Napoca, who guided me patiently in the field of fixed point theory from the very beginning of my MSc Dissertation, continuing with the research included in my PhD Thesis, and extended even today. I take this opportunity to thank him heartedly.

It is impossible to acknowledge individually colleagues and friends to whom I am indebted for support in writing this monograph. I must, however, express my appreciation and thanks to Acad. Petar Kenderov from the Institute of Mathematics, Bulgarian Academy of Sciences, Sofia, to Dr. Jaime Zavala Carvajal, Pontificia Universidad Catolica de Valparaiso, Chile, to Dr. Peter Kortesi from the University of Miskolc, Hungary and to Dr. Goetz Pfeiffer, from National University of Ireland in Galway, for the excellent conditions they offered me during my visits at their institutions, when some parts of this book have been written and various bibliographical references were provided to me.

I am also indebted to many scientists whose research work formed a basis for this monograph. I wish to express my thanks to all of them, and to each in a measure proportional to my indebtedness. Amongst them, I particularly want to thank Professor B.E. Rhoades from Indiana University, U.S.A., who has sent me the reprints of his considerable and long term work in the field of approximating fixed points.

Last but not most of all, I would like to express my deepest gratitude to Zoița, my wife, for her patient support and insistent pushing me toward my desk in order to finish the book, as well as to Mădălina and Ruxandra, my daughters, who contributed directly and in different manners to the accomplishment of this book.

Baia Mare

Vasile BERINDE  
January 18, 2002

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## Introduction

A possible starting point in judging the merits of the book would be the idea that it is a drop in an ocean of intensive and extensive research work. Consequently, our aim was to present, as clearly and completely as possible, a survey of the basic results in iterative approximation of fixed points.

In order to meet the taste of the majority of scientists interested in this area, our intention was to produce an in-depth and up-to-date coverage of about 400 recent publications out of more than 1575 entries in the reference list. However, it would have been impossible to cover consistently the diversity of research work that has been done in the field of iterative approximation of fixed points and related areas.

The diversity of results on this topic comes mainly from three directions:

1. The variety of the underlying spaces where the operators are defined;
2. The variety of contractiveness assumptions and/or topological properties associated with these operators;
3. The variety of assumptions on the parameters that define a certain fixed point iteration procedure. Sometimes these parameters depend also on the geometry of the ambient space and/or on the properties of the considered operator.

Therefore, the author is perfectly aware of the risks he has taken when designing the book. It is doubtful that the structure, contents and organization of the material in each Chapter or Section will meet all the needs and horizons of the specialists working in this area.

As a general rule, emphasis is put only on some generic results regarding the main topic, since it would be impossible to aim for complete coverage. Usually, for each iterative fixed point procedure, some of the most interesting, representative and significant results are completely presented, while some others are formulated as exercises or are only briefly mentioned.

Moreover, in some chapters and sections we did not always include the most general result related to a certain topic, but the most accessible one. In these circumstances, we tried to stress on the most clear result when possible and to mention the other more general results. Simultaneously we tried to illustrate the diversity of the results, and so to avoid presenting the convergence results of different iterative processes in the same or in a similar setting.

No matter how narrow its topic, a book cannot be written in a self-contained manner when space limits are imposed. This is the reason why we preferred to include some (auxiliary) results without (detailed) proofs, and to insert other much more diversified results instead. The readers interested in knowing the details should consult the appropriate references, as the bibliography, with its more than 1575 references, provides additional sources of results and approaches on the approximation of fixed points.

In order to make reading as fluent as possible, we generally tried to avoid bibliography citations in the text of the sections. Instead we have supplemented each chapter with a special section containing a set of “Bibliographical Comments”, where many literature citations are given and other related results are sometimes mentioned. Including a result in a certain section does not mean it is the most general in that area: in several circumstances the taste of the author was simply the dominant reason, when we tried to mention the similar more general or most important results.

Despite the considerable amount of overlapping research work on the Ishikawa and Mann iteration procedures, we however decided to have a distinct chapter for each one, where specific results were also included. Apart from the sections “Exercises and Miscellaneous results”, in some sections at least one proof or parts of the proof are left for the reader to be completed.

Throughout the book we adopted the following numbering system: in each Chapter the Definitions, Lemmas and Theorems are numbered using two digits, while the equations are numbered using one digit only. For example, Theorem 3.6 or Definition 4.5 or Lemma 7.2 denote the sixth theorem included in Chapter 3, the fifth definition in Chapter 4 and the second lemma in Chapter 7, respectively. When references to them are needed, examples are also numbered, in the same described manner. On the contrary, when referring to a certain equation we shall say, for example, equation (3) in Chapter 4 instead of equation (4.3).

In writing non-English author names, we ignored the specific diacritical signs. So, Hadžić and Păvăloiu will be written simply as Hadzic and Pavaloiu, respectively. For Krasnosel’skij we preferred the form Krasnoselskij, even though in some sources other variants (e.g., Krasnoselskii) can be found.

Concluding the introduction, we want to stress on the main merit of this book: the very fact that it was written down. However, we hope that, by gathering and systematizing various significant results in the dynamic field of fixed point iteration procedures, we provide a useful tool for many postgraduate and PhD students as well as for any interested researchers.