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Hirotaka Akiyoshi · Makoto Sakuma
Masaaki Wada · Yasushi Yamashita

Punctured Torus Groups and 2-Bridge Knot Groups (I)

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Authors

Hirotaka Akiyoshi
Osaka City University
Advanced Mathematical Institute
3-3-138 Sugimoto, Sumiyoshi-ku,
Osaka 558-8585
Japan
e-mail: akiyoshi@sci.osaka-cu.ac.jp

Makoto Sakuma
Department of Mathematics
Hiroshima University
1-3-1, Kagamiyama
Higashi-Hiroshima 739-8526
Japan
e-mail: sakuma@math.sci.hiroshima-u.ac.jp

Masaaki Wada
Yasushi Yamashita
Department of Information
and Computer Sciences
Nara Women's University
Kita-uoya Nishimachi
Nara 630-8506
Japan
e-mail: wada@ics.nara-wu.ac.jp
yamasita@ics.nara-wu.ac.jp

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Preface

The main purpose of this monograph is to give a full description of Jorgensen's theory on the space $\mathcal{QF}$ of quasifuchsian (once) punctured torus groups with a complete proof. Our method is based on Poincare's theorem on fundamental polyhedra. This geometric approach enabled us to extend Jorgensen's theory beyond the quasifuchsian space and apply to knot theory.

1. History

By the late 70's Troels Jorgensen had made a series of detailed studies on the space $\mathcal{QF}$ of quasifuchsian (once) punctured torus groups from the view point of their Ford fundamental domains. These studies are summarized in his famous unfinished paper [40]. In it, he gave a complete description of the combinatorial structure of the Ford domain of every quasifuchsian punctured torus group, and showed that the space $\mathcal{QF}$ can be described in terms of the combinatorics of the faces of the Ford domain. This led to the description of $\mathcal{QF}$ in terms of the Farey triangulation, or the modular diagram. As a byproduct, the first examples of surface bundles over the circle with complete hyperbolic structures were obtained (cf. [41] and [43]).

To date, most of Jorgensen's work has not been published, yet it became widely known, motivated various research projects, and was successfully applied. His work, together with Riley's construction [67] of the complete hyperbolic structure on the figure-eight knot complement, has motivated Thurston's uniformization theorem of surface bundles over the circle [77] (cf. [63]). It had also motivated the experimental study by Mumford, McMullen and Wright [60] of the limit sets of quasifuchsian punctured torus groups. This work was sublimated into the beautiful book [61] by Mumford, Series and Wright, which displays deeply hidden fractal shapes of the space $\mathcal{QF}$ and the limit sets of punctured torus Kleinian groups.

2. Motivation

The authors' interest in Jorgensen's work grew from knot theory. We are interested in hyperbolic knots, and more generally hyperbolic links, i.e., mutually disjoint circles embedded in the 3-sphere S^3 whose complements admit complete hyperbolic structures of finite volume. Recall that the Ford domain of a complete cusped hyperbolic manifold of finite volume is the geometric dual to the canonical ideal polyhedral decomposition introduced by Epstein and Penner [27] (cf. [81]). Thus, by virtue of Mostow rigidity, the combinatorial structure of the Ford domain is a complete invariant of the topological type of such a manifold. In particular, by the knot complementary theorem due to Gordon and Luecke [32], this gives a complete invariant of a hyperbolic knot. In the joint work [71] with Weeks, the second author gave certain topological decompositions of 2-bridge link complements into topological ideal tetrahedra, by imitating Jorgensen's decomposition of punctured torus bundles over the circle (cf. [29]), and conjectured that they are combinatorially equivalent to the canonical decompositions. Here, a 2-bridge link is a link which can be drawn with only two local maxima and minima in the vertical direction (see Fig. 0.1). We had thought that if we could understand Jorgensen's work, then we would be able to prove the conjecture.

3. Extending of Jorgensen's theory beyond the quasifuchsian space and application to 2-bridge links

Fortunately, this turned out to be the case. Namely, we found a very natural way to understand the hyperbolic structures and the canonical decompositions of the 2-bridge link complements in the context of Jorgensen's work. To describe the idea, recall that the 2-bridge links are parametrized by pairs (p, q) of relatively prime integers (see [22, Chap. 12]) and that the complement of the 2-bridge link of type (p, q) is homeomorphic to (the interior of) the manifold obtained from $S \times [-1, 1]$, with S a 4-times punctured sphere, by attaching 2-handles along $\alpha \times (-1)$ and $\beta \times 1$, where α and β are simple loops on S of slopes $1/0$ and q/p, respectively (see Sect. 2.1, p. 16, for the definition of a slope); in particular, the link group (i.e., the fundamental group of the complement of the link) is isomorphic to the quotient group $\pi_1(S)/\langle\langle \alpha, \beta \rangle\rangle$, where $\langle\langle \cdot \rangle\rangle$ denotes the normal closure. The extended Jorgensen's theory realizes the operation of attaching 2-handles by a continuous family of hyperbolic cone-manifolds, whose cone axes are the union of the *upper* and *lower tunnels*, i.e., the co-cores of the 2-handles (see Fig. 0.1).

According to Keen-Series' theory of pleating varieties [44, 45, 46, 47, 49], $\mathcal{QF}$ is foliated by the pleating varieties, $\mathcal{P}(\lambda^-, \lambda^+)$, where (λ^-, λ^+) runs over (ordered) pairs of distinct projective measured laminations of the punctured torus T. By extending Jorgensen's theory beyond the quasifuchsian space (cf.

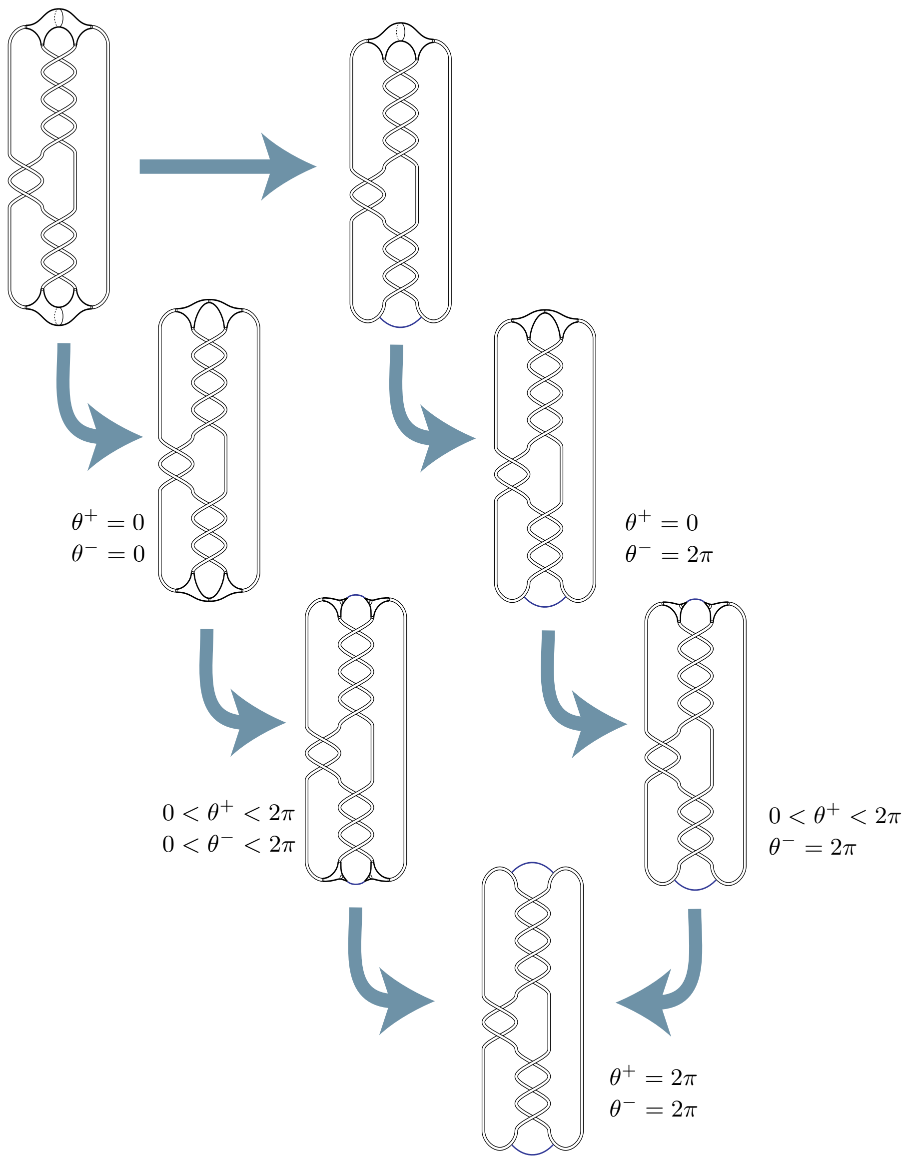

Fig. 0.1. Continuous family of hyperbolic cone-manifolds $M(\theta^-, \theta^+)$

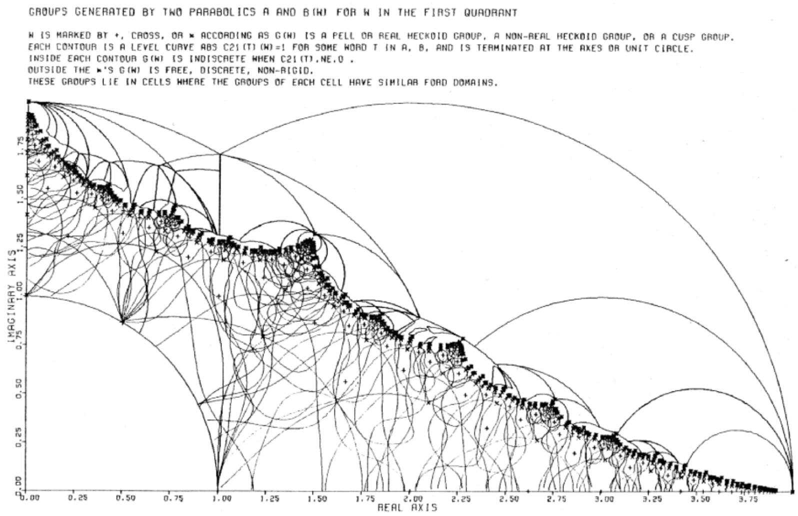

Fig. 0.2a. Riley's pioneering exploration of groups generated by two parabolic transformations. This computer-drawn picture has been circulated among the experts and has inspired many researchers in the fields of Kleinian groups and knot theory. This specific copy of the picture was obtained directly from Prof. Riley by the third author when he visited SUNY Binghamton in February 1991.

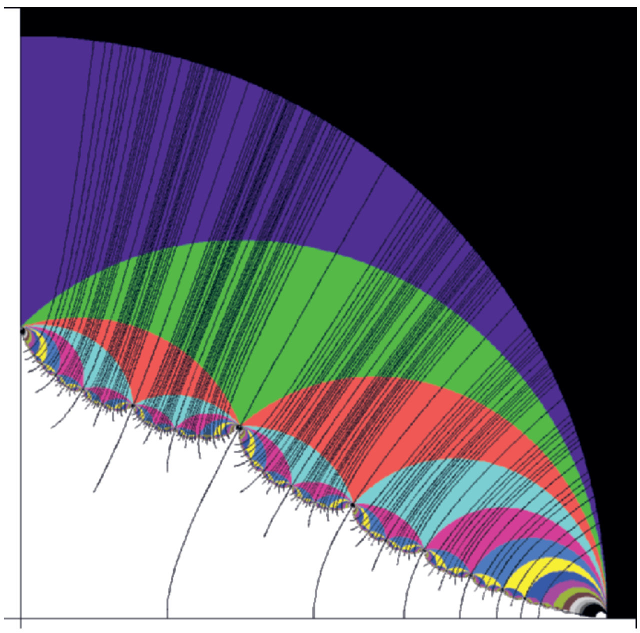

Fig. 0.2b. Riley slice of Schottky space together with pleating rays and their extensions

[66]), we found that if (λ^-, λ^+) is rational, i.e., each of $\lambda^\pm$ are rational, then the following hold:

1. The pleating variety $\mathcal{P}(\lambda^-, \lambda^+)$ has a natural extension to the outside of $\mathcal{QF}$ in the space of type-preserving representations of the fundamental group $\pi_1(T)$.
2. Each point in the extension is the holonomy representation of a certain hyperbolic cone manifold, which is commensurable with the hyperbolic cone manifold, $M(\theta^-, \theta^+)$, whose underlying space is the complement of a 2-bridge link and whose singular set is the union of the upper and lower tunnels, which have the cone angles θ^+ and θ^-, respectively. Moreover the 2-bridge link is of type (p, q), or of slope q/p, if (λ^-, λ^+) is equivalent to $(1/0, q/p)$ by a modular transformation.
3. If the (edge path) distance $d(1/0, q/p)$ in the Farey triangulation is ≥ 3, namely if $q \not\equiv \pm 1 \pmod p$, then the hyperbolic cone manifold $M(\theta^-, \theta^+)$ exists for every pair of cone angles in $[0, 2\pi]$. Thus we have a continuous family of hyperbolic cone manifolds connecting $M(0, 0)$, the quotient hyperbolic manifold of a doubly cusped group, with $M(2\pi, 2\pi)$, the complete hyperbolic structure of the 2-bridge link complement.
4. If $1 \leq d(1/0, q/p) \leq 2$, namely if $q \equiv \pm 1 \pmod p$ and $p \neq 0$, then the hyperbolic cone manifold $M(\theta^-, \theta^+)$ exists for every pair of cone angles in $[0, 2\pi]$, except the pair $(2\pi, 2\pi)$. In addition, if $p \geq 3$, $M(\theta^-, \theta^+)$ collapses to the base orbifold of the Seifert fibered structure of the link complement as both cone angles approach 2π.
5. The holonomy group of $M(\theta^-, \theta^+)$ is discrete if and only if $\theta^\pm \in \{2\pi/n \,|\, n \in \mathbb{N}\} \cup \{0\}$. In particular, that of $M(2\pi, 2\pi/n)$ is generated by two parabolic transformations, which Riley called a *Heckoid group* in [68].

Actually, we have constructed these hyperbolic cone manifolds by explicitly constructing "Ford fundamental polyhedra". In other words, we have extended Jorgensen's description of the Ford fundamental polyhedra for quasifuchsian punctured torus groups to those of the hyperbolic cone manifolds arising from the 2-bridge links. In particular, we have shown that the canonical decompositions of hyperbolic 2-bridge link complements are isotopic to the topological ideal tetrahedral decompositions constructed in [71], proving the conjecture which motivated our project.

The above result also enables us to locate the 2-bridge link groups in the representation space (Fig. 0.2b). The shaded region of the figure illustrates (the first quadrant of) the *Riley slice* of the Schottky space, i.e., the subspace of $\mathbb{C}$ consisting of those complex numbers ω such that the group

$$G_\omega = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \right\rangle$$

is discrete and free and such that the quotient $\Omega(G_\omega)/G_\omega$ of the domain of discontinuity is homeomorphic to the 4-times punctured sphere S (Definition

5.3.5). Each shaded region represents groups whose Ford domains have the same combinatorics. The lines in the shaded region are pleating rays of the Riley slice ([45]) and their extensions to the outside of the Riley slice correspond to the hyperbolic cone-manifolds $M(2\pi, \theta)$. In particular the endpoints with positive imaginary parts represent hyperbolic 2-bridge link groups and those on the real line represent the orbifold fundamental groups of the base orbifolds of the Seifert fibered structures of non-hyperbolic 2-bridge link complements.

We think this realizes what Riley had in mind, for he devoted time and effort to identify 2-bridge link groups in the space of non-elementary two parabolic groups, yielding the mysterious output in Fig. 0.2a ([69]).

This describes a relation between the hyperbolic structure and the bridge structure of a 2-bridge link complement. Since a bridge structure is a kind of Heegaard structure, it is naturally expected that a similar relation holds between the hyperbolic structures and the Heegaard structures of hyperbolic manifolds. In particular, we conjecture that this is the case for tunnel number 1 hyperbolic knots and their unknotting tunnels. An *unknotting tunnel* for a knot K is an arc τ in S^3 with $\tau \cap K = \partial\tau$ such that the complement of an open regular neighborhood is homeomorphic to a genus 2 handlebody. A knot which admits an unknotting tunnel is said to have *tunnel number* 1. For example, a 2-bridge knot has tunnel number 1 and each of the upper and lower tunnels is an unknotting tunnel. Tunnel number 1 knots have been extensively studied, and in particular, non-hyperbolic tunnel number 1 knots were classified by [59]. An unknotting tunnel τ of a tunnel number 1 knot K gives a *Heegaard structure* of the knot complement $S^3 - K$, in the sense that $S^3 - K$ is homeomorphic to (the interior) of the manifold obtained from the genus 2 handlebody by adding a 2-handle, where τ corresponds to the co-core of the 2-handle. We would like to propose the following conjecture.

Conjecture. Let K be a tunnel number 1 hyperbolic knot and let τ be an unknotting tunnel for K. Then there is a continuous family of hyperbolic cone manifolds whose underlying space is the knot complement and whose cone axis is the unknotting tunnel τ, where the cone angle varies from 0 to 2π. In particular, τ is isotopic to a geodesic in the hyperbolic manifold $S^3 - K$.

4. Related results

Some of these results were announced in [8, 9, 10], and our original plan was to write a single paper or a book which contains the whole story. However, we found it very difficult to explain the whole theory at once, and thus decided to divide it into a few papers. This monograph is the first part of the series, and its main purpose is to give a full description of Jorgensen's theory on the space $\mathcal{QF}$ with a complete proof.

For Jorgensen's theory on the space $\mathcal{QF}$, supervised by Dunfield and partially influenced by [9] and [78], Schedler [72] gave a treatment based on the

theory of holomorphic motions. Though the bijectivity of the *side parameter* map is not proved in his paper, his approach using holomorphic motions is natural and further development is expected in the future.

Our approach in turn is based on Poincare's theorem on fundamental polyhedra. This geometric approach enables us to extend Jorgensen's theory beyond the quasifuchsian space, where we need to treat indiscrete groups.

For (attempts of) expositions of Jorgensen's theory without proof, see [75, 8, 9, 65, 70].

The first author has extended Jorgensen's theory to the closure of $\mathcal{QF}$ in [2]. In particular, a rigorous proof was given to the well-known description of the Ford domain of the punctured torus bundles over the circle (cf. [12, 64]). We note that Lackenby [52] gave a topological proof to the fact that Jorgensen's ideal triangulations of punctured torus bundles are genuine geometric decompositions. Gueritaud [33] also gave an alternative proof to this fact by using the angle structure. In the appendix of the paper, Futer proves by modifying Gueritaud's argument that the topological ideal triangulations of the 2-bridge link complements in [71] are also geometric. Moreover, Gueritaud [34] also proved that these geometric decompositions are canonical.

In [3], the first author has found a nice relation between Jorgensen's parameter of $\mathcal{QF}$ and the conformal end invariant of elements of $\mathcal{QF}$. This together with Brock's results [21] leads to an estimate of the convex core volume in terms of Jorgensen's parameter. He has also found interesting applications of Jorgensen's theory to knot theory in [4].

The computer program, OPTi [78] (cf. [79]), has been developed by the third author for the project, and it has been a driving force for our work. It is our pleasure that it has now become a favorite tool for various colleagues in the world.

Collaborating with Komori and Sugawa, the third and last authors launched a project to draw Bers' slices of $\mathcal{QF}$, and various mysterious pictures have been produced ([50] and [82]).

5. A quick trip through Jorgensen's theory and its generalization

Jorgensen's theory enables us to intuitively understand how a simple fuchsian group evolves into complicated quasifuchsian punctured torus groups and boundary groups, by looking at their Ford domains (see Figs. 0.3–0.10, 0.17, 0.19–0.21 and 1.2). Jorgensen expresses this phenomenon as follows. *The Ford domain records the history of how the quasifuchsian group evolved from a simple fuchsian group.*

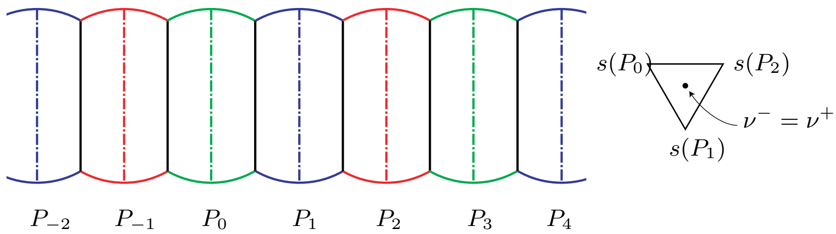

Fig. 0.3. $(1/3, 1/3, 1/3)$

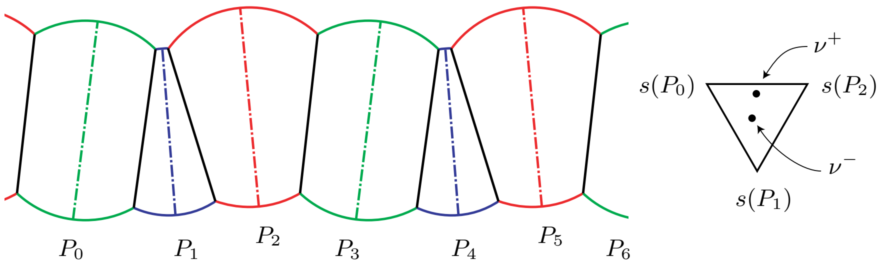

Fig. 0.4. $(0.421397 - 0.0483593\mathrm{i}, 0.295605 - 0.0422088\mathrm{i}, 0.282998 + 0.0905681\mathrm{i})$

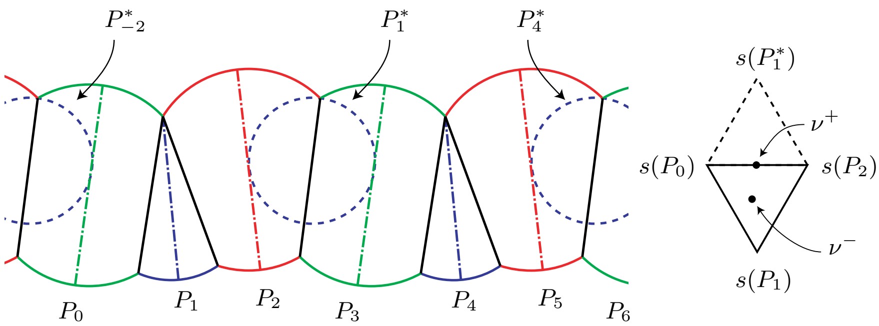

Fig. 0.5. $(0.433791 - 0.0551654\mathrm{i}, 0.290295 - 0.0481496\mathrm{i}, 0.275914 + 0.103315\mathrm{i})$

5.1. A fuchsian punctured torus group

The starting point of Jorgensen's theory is the fuchsian group illustrated in Fig. 0.11. For each integer j, let L_j be the geodesic in the upper half plane model $\mathbb{H}^2$ of the hyperbolic plane, represented by the Euclidean half circle with center $j/3$ and radius $1/3$. Let P_j be the order 2 elliptic transformation whose fixed point is equal to the highest point $(j+\mathrm{i})/3$ of L_j where $\mathrm{i} = \sqrt{-1}$. Then P_j interchanges the inside and outside of L_j and acts on L_j as a Euclidean isometry. The product $P_{j+2}P_{j+1}P_j$ is equal to the parabolic transformation $K(z) = z + 1$. Note that $P_{j+3n} = K^n P_j K^{-n}$ for every j, $n \in \mathbb{Z}$. Let Γ be

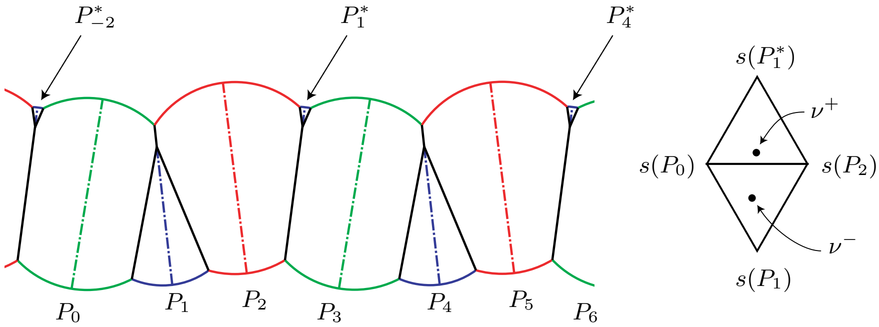

Fig. 0.6. $(0.444228 - 0.0608968\mathrm{i}, 0.285823 - 0.0531522\mathrm{i}, 0.269949 + 0.114049\mathrm{i})$

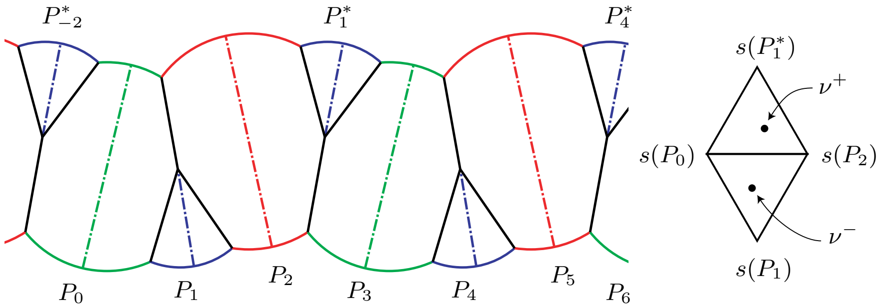

Fig. 0.7. $(0.496414 - 0.0895542\mathrm{i}, 0.263465 - 0.0781648\mathrm{i}, 0.240121 + 0.167719\mathrm{i})$

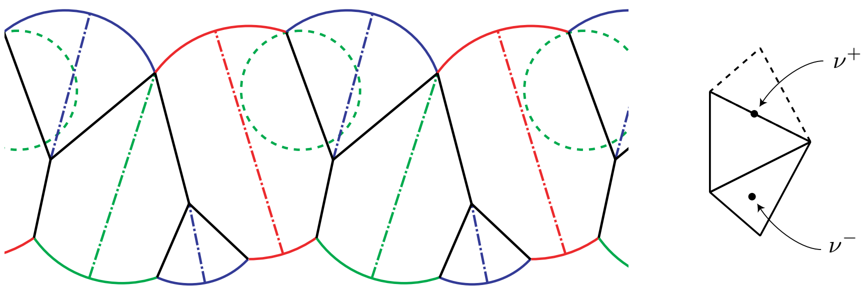

Fig. 0.8. $(0.549741 - 0.118838\mathrm{i}, 0.240619 - 0.103725\mathrm{i}, 0.20964 + 0.222563\mathrm{i})$

the group generated by $\{P_j \mid j \in \mathbb{Z}\}$. Then it is generated by three successive elements, say P_0, P_1 and P_2. Consider the shaded region R in Fig. 0.11. Then the edges of R are paired by P_0, P_1, P_2 and K. By applying Poincare's theorem on fundamental polyhedra to this setting, we see that R is a fundamental domain of the group Γ and

$$\Gamma \cong \langle P_0, P_1, P_2 \mid P_0^2 = P_1^2 = P_2^2 = 1 \rangle \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}).$$

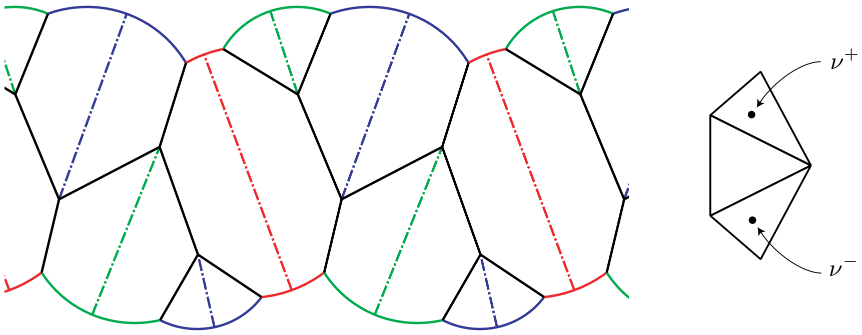

Fig. 0.9. $(0.594262 - 0.143287i, 0.221545 - 0.125063i, 0.184193 + 0.26835i)$

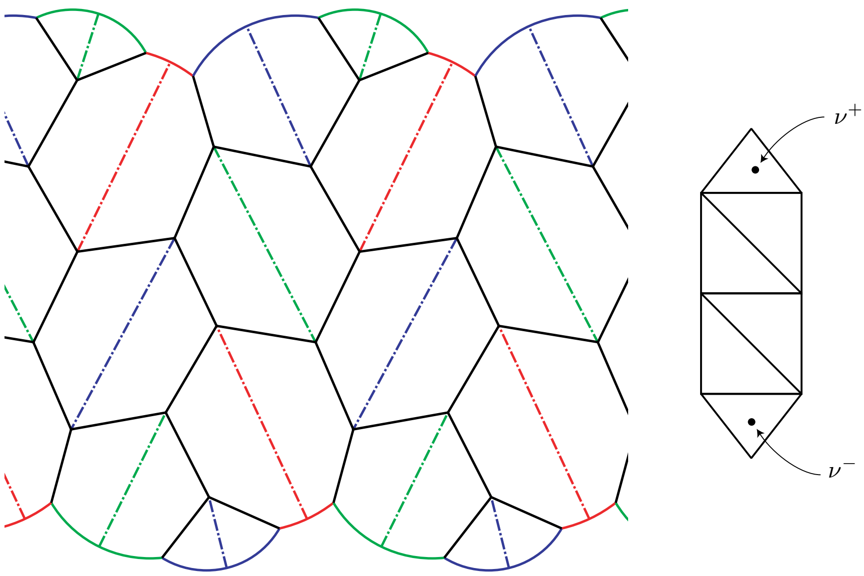

Fig. 0.10. $(0.652971 - 0.175526i, 0.196392 - 0.153203i, 0.150637 + 0.328729i)$

As shown in Fig. 0.12, the quotient $\mathbb{H}^2/\Gamma$ is a hyperbolic orbifold, $\mathcal{O}$, with underlying space once-punctured sphere and with three cone points of cone angle π. The subgroup Γ_0 of Γ of index 2, obtained as the kernel of the homomorphism $\Gamma \to \mathbb{Z}/2\mathbb{Z}$ sending each generator P_j to the generator of $\mathbb{Z}/2\mathbb{Z}$, is a rank 2 free group generated by $A := KP_0 = P_2P_1$ and $B := K^{-1}P_2 = P_0P_1$. The union $R \cup K(R)$ is a fundamental domain of Γ_0, and the quotient $\mathbb{H}^2/\Gamma_0$ is homeomorphic to the once-punctured torus, T, where the puncture corresponds to the commutator $[A, B] = K^2$. Thus Γ_0 is a *fuchsian*

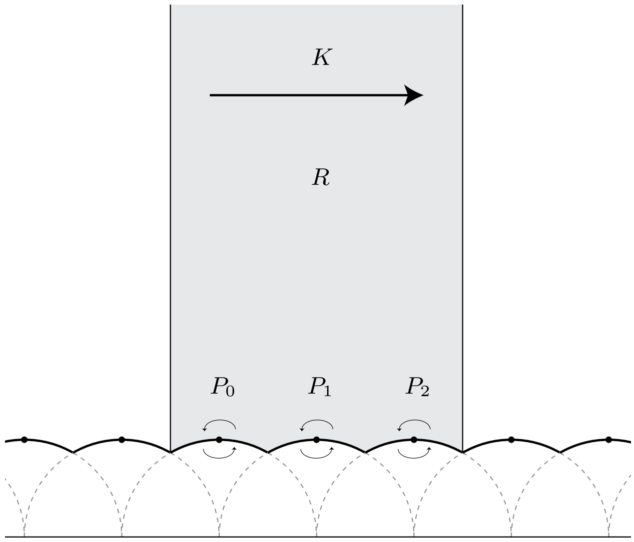

Fig. 0.11. Fuchsian group $\Gamma = \langle P_0, P_1, P_2 \rangle$ and its fundamental region R

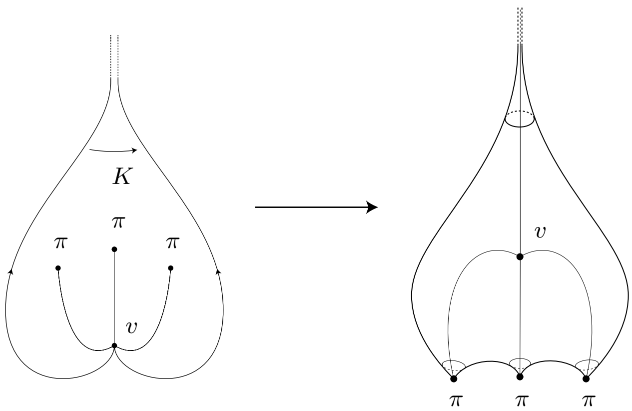

Fig. 0.12. By applying the edge pairings P_0, P_1 and P_2 to the fundamental region R, we obtained the surface on the left hand side. By further applying the edge pairing K to this surface, we obtain the orbifold $\mathcal{O}$ with underlying space once-punctured sphere and with three cone points of cone angle π.

punctured torus group, i.e., it is a discrete free group generated by two elements with parabolic commutator. It is well-known that every fuchsian punctured torus group has a $\mathbb{Z}/2\mathbb{Z}$-extension with quotient homeomorphic to $\mathcal{O}$ as a topological orbifold (cf. [40, Sect. 2]). Thus we abuse terminology and call the extended group a *fuchsian punctured torus group.*

Now look at the region, F, exterior to all L_j. Then this is a "fundamental domain of Γ (resp. Γ_0) modulo $\langle K \rangle$ (resp. $\langle K^2 \rangle$)" (cf. Proposition 1.1.3). This region is called the *Ford polygon* of Γ (resp. Γ_0). This can be regarded as the "Dirichlet domain of Γ centered at ∞", because

$$F = \{x \in \mathbb{H}^2 \mid d(x, H_\infty) \leq d(x, ZH_\infty) \text{ for every } Z \in \Gamma\},$$

where H_∞ is a sufficiently small horodisk centered at ∞. This implies that the image of ∂F in $\mathbb{H}^2/\Gamma$ is equal to the *cut locus* of $\mathbb{H}^2/\Gamma$ with respect to the cusp, i.e., the set of points of $\mathbb{H}^2/\Gamma$ which has more than two shortest geodesics to the cusp. See Proposition 5.1.3, for a description of the Ford polygons of general fuchsian punctured torus groups.

5.2. 3-dimensional picture of the fuchsian punctured torus group

Figure 0.3 gives a 3-dimensional picture of the group Γ in Fig. 0.11. The elliptic transformation P_j acts on the upper half space model $\mathbb{H}^3$ of the hyperbolic 3-space as the π-rotation around the geodesic joining the two points $(j \pm \mathrm{i})/3$, where $\mathrm{i} = \sqrt{-1}$. (Here we identify the complex plane $\mathbb{C}$ with the boundary of the closure $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{C}$.) The *isometric circle*

$$I(P_j) = \{z \in \mathbb{C} \mid |P_j'(z)| = 1\}$$

has center $c(P_j) = j/3$ and radius $1/3$. The hyperplane of $\mathbb{H}^3$ bounded by the isometric circle $I(P_j)$ is called the *isometric hemisphere* of P_j and is denoted by $Ih(P_j)$. Then P_j interchanges the exterior $Eh(P_j)$ and the interior $Dh(P_j)$ of the isometric hemisphere $Ih(P_j)$, and acts on $Ih(P_j)$ as a Euclidean isometry. By the argument in Subsection 5.1, we see that the common exterior $\cap_j Eh(P_j)$, where j runs over $\mathbb{Z}$, is a "fundamental domain of the action of Γ (resp. Γ_0) on $\mathbb{H}^3$ modulo $\langle K \rangle$ (resp. $\langle K^2 \rangle$)". Thus it is equal to the *Ford domain* $Ph(\Gamma)$ of Γ, which is defined to be the common exteriors of the isometric hemispheres of all elements of Γ that do not fix ∞ (see Definition 1.1.2 and Proposition 1.1.3). As in the previous subsection, the Ford domain can be regarded as the "Dirichlet domain of Γ centered at ∞", namely

$$Ph(\Gamma) = \{x \in \mathbb{H}^3 \mid d(x, H_\infty) \leq d(x, ZH_\infty) \text{ for every } Z \in \Gamma\},$$

where H_∞ is a sufficiently small horoball centered at ∞. Thus the image of $\partial Ph(\Gamma)$ in $\mathbb{H}^3/\Gamma \cong \mathcal{O} \times (-1, 1)$ is equal to the *cut locus* of $\mathbb{H}^3/\Gamma$ with respect

to the cusp, i.e., the set of points of $\mathbb{H}^3/\Gamma$ which has more than two shortest geodesics to the cusp: We call it the *Ford complex* of Γ.

Let $\overline{Ph}(\Gamma)$ be the closure of the Ford domain $Ph(\Gamma)$ in $\overline{\mathbb{H}}^3$. Then the intersection $P(\Gamma) := \overline{Ph}(\Gamma) \cap \mathbb{C}$ has precisely two connected components, the *upper polygon* $P^+(\Gamma)$ and the *lower polygon* $P^-(\Gamma)$. $P^\pm(\Gamma)$ is a fundamental domain of the action of Γ on $\Omega^\pm(\Gamma)$ modulo $\langle K \rangle$, where $\Omega^+(\Gamma)$ and $\Omega^-(\Gamma)$ are the upper and lower components of the domain of discontinuity $\Omega(\Gamma) = \mathbb{C} - \mathbb{R}$.

5.3. Parameters for punctured torus groups

Jorgensen's theory describes what happens to the Ford domain when we deform the group Γ keeping the condition that $K = P_2P_1P_0$ is the parabolic transformation $z \mapsto z+1$, or equivalently, deform the group Γ_0 keeping the condition that $[A, B]$ is the parabolic transformation $z \mapsto z+2$. Thus we first need to describe the space of all such groups. Let $\mathcal{X}$ be the space of the equivalence classes of marked subgroups Γ of $PSL(2,\mathbb{C})$ generated by an *elliptic generator triple* (P_0, P_1, P_2), i.e., a triple of order 2 elliptic transformations P_0, P_1 and P_2 such that the product $K := P_2P_1P_0$ is a parabolic transformation (cf. Definition 2.1.1). Two such marked groups Γ and Γ', endowed with elliptic generator triples (P_0, P_1, P_2) and (P_0', P_1', P_2'), respectively, are *equivalent* if they are conjugate in $PSL(2,\mathbb{C})$ and if the conjugation maps (P_0, P_1, P_2) to (P_0', P_1', P_2') (cf. Definition 2.2.6). We do not distinguish between an element of $\mathcal{X}$ and its representative. The space $\mathcal{X}$ is identified with a quotient of a subspace of the cartesian product $PSL(2,\mathbb{C})^3$ and thus is endowed with the quotient topology.

By a *marked punctured torus group*, we mean an element Γ of $\mathcal{X}$ such that Γ is discrete and isomorphic to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$.

Then $\mathcal{X}$ is identified with the space of the equivalence classes of the non-trivial *Markoff triples*. Here a Markoff triple is a triple $(x, y, z) \in \mathbb{C}^3$ satisfying the Markoff equation

$$x^2 + y^2 + z^2 = xyz.$$

It is non-trivial if it is different from $(0, 0, 0)$. Two Markoff triples (x, y, z) and (x', y', z') are equivalent if the latter is equal to (x, y, z), $(x, -y, -z)$, $(-x, y, -z)$ or $(-x, -y, z)$. We associate to each marked group $\Gamma \in \mathcal{X}$, endowed with an elliptic generator triple (P_0, P_1, P_2), the equivalence class of a Markoff triple (x, y, z) by the following rule.

$$(x, y, z) = (\mathrm{tr}(KP_0), \mathrm{tr}(KP_1), \mathrm{tr}(KP_2)) = (\mathrm{tr}(A), \mathrm{tr}(AB), \mathrm{tr}(B)),$$

where $A = KP_0$ and $B = K^{-1}P_2$. Note that the right hand side is defined only up to sign, because it depends on the lifts to $SL(2,\mathbb{C})$. If we take the lifts so that the lift of AB is the product of the lift of A and that of B, then (x, y, z) satisfies the Markoff equation and its equivalence class is uniquely

determined by the equivalence class of the marked group Γ, and vice versa (Propositions 2.3.6 and 2.4.2). For example, the group in Fig. 0.11 corresponds to the Markoff triple $(3, 3, 3)$.

More geometric parameter of (a subspace of) $\mathcal{X}$ is the *complex probability* defined as follows. Let (x, y, z) be a Markoff triple such that none of the entries are zero. Then its equivalence class is completely determined by the triple $(a_0, a_1, a_2) \in (\mathbb{C} - \{0\})^3$ defined by

$$(a_0, a_1, a_2) = (\frac{x}{yz}, \frac{y}{zx}, \frac{z}{xy}).$$

The only constraint on this triple is the identity

$$a_0 + a_1 + a_2 = 1,$$

and thus this triple is called a *complex probability*. This parameter has the geometric meaning that each entry gives the difference between the centers of the isometric circles of the elliptic generators. Namely,

$$a_0 = c(P_2) - c(P_1), \quad a_1 = c(P_3) - c(P_2), \quad a_2 = c(P_4) - c(P_3) = c(P_1) - c(P_0).$$

Here $P_3 = KP_0K^{-1}$ and $P_4 = KP_1K^{-1}$. Moreover, there is a nice geometric construction of a marked group from the corresponding complex probability (see Sect. 2.4, p. 29). The complex probability for the marked group in Fig. 0.3, for example, is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The triples in the captions of Figs. 0.3–0.10, 0.17, 0.19–0.25 are the complex probabilities of the corresponding marked punctured torus groups.

5.4. Small deformation of the fuchsian punctured torus group

Now let us study what happens to the Ford domain if we deform the group Γ in Fig. 0.3 a little in the space $\mathcal{X}$, namely we perturb the complex probability from $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ a little. The answer is that nothing happens to the combinatorial structure of the Ford domain (see Fig. 0.4). Namely, the polyhedron $\cap_j Eh(P_j)$ continues to be the Ford domain of the (deformed) group Γ. This fact can be proved by using Poincare's theorem on fundamental polyhedra and the following facts for the (deformed) group Γ, which are consequences of the "chain rule for isometric circles" (Lemmas 4.1.2 and 4.1.3).

1. Each face $Ih(P_j) \cap (\cap_j Eh(P_j))$ of the polyhedron $\cap_j Eh(P_j)$ is symmetric with respect to P_j.
2. The sum of the dihedral angles of the polyhedron $\cap_j Eh(P_j)$ along any three successive edges $Ih(P_j) \cap Ih(P_{j+1})$ is equal to 2π.

The above conclusion on the Ford domain corresponds to a special case of Proposition 6.2.1 (Openness), whose rigorous proof is given in Chap. 7. We note that Schedler [72] explains this phenomenon by developing a general theory of Ford domains based on the theory of holomorphic motions. A key fact behind both proofs is that $Ph(\Gamma)$ has no *hidden isometric hemispheres*, that is, for every element A of Γ which does not fix ∞, $\overline{Ph}(\Gamma) \cap \overline{Ih}(A) \neq \emptyset$ only if $Ih(A)$ support a 2-dimensional face of the polyhedron $\cap_j Eh(P_j)$, i.e., $A = P_j$ for some j for the group in Fig. 0.4 (see Lemma 7.1.6). Here $\overline{Ih}(A)$ denotes the closure of $Ih(A)$ in $\overline{\mathbb{H}}^3$.

The deformed group Γ is a marked *quasifuchsian punctured torus group*, i.e., it is a quasi-conformal deformation of the fuchsian punctured torus group in Fig. 0.3, or equivalently, the limit set of Γ continues to be a Jordan circle in the Riemann sphere $\partial\mathbb{H}^3$ and the quotient $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ continues to be homeomorphic to the product $\mathcal{O} \times [-1, 1]$.

The *quasifuchsian punctured torus space* $\mathcal{QF}$ is the subspace of $\mathcal{X}$ consisting of all (marked) quasifuchsian punctured torus groups. $\mathcal{QF}$ is an open subset of the 2-dimensional complex manifold $\mathcal{X}$. Bers' simultaneous uniformization theorem says that a quasifuchsian punctured torus group Γ is uniquely determined by the pair $(\Omega^-(\Gamma_0)/\Gamma_0, \Omega^+(\Gamma_0)/\Gamma_0)$ of punctured torus Riemann surfaces (see, for example, [38] or [55]). This correspondence implies a holomorphic isomorphism between the quasifuchsian space $\mathcal{QF}$ and the product $\mathrm{Teich}(T) \times \mathrm{Teich}(T) \cong \mathbb{H}^2 \times \mathbb{H}^2$ of the Teichmuller space of T. Jorgensen's theory also gives yet another parameterization of $\mathcal{QF}$ in terms of $\mathbb{H}^2 \times \mathbb{H}^2$ (see Main Theorem 1.3.5).

Though the space $\mathcal{QF}$ itself has a simple structure, its location in $\mathcal{X}$ is very complicated. Jorgensen's theory enables us to plot the shape of $\mathcal{QF}$. See Fig. 0.13 illustrating a slice of $\mathcal{QF}$ in $\mathcal{X}$. This is an output of OPTi [78], which in turn is based on Jorgensen's theory. See also the beautiful pictures in [61].

5.5. Birth of a new face in the Ford domain

We can continue the deformation in the previous subsection until some of the circular edges of the upper/lower polygons $P^{\pm}(\Gamma)$ shrinks to a point (see Fig. 0.5). Assume for simplicity that the circular edge $I(P_1) \cap P^+(\Gamma)$ of the upper polygon $P^+(\Gamma)$ shrinks to a point, v. (The existence of such a deformation is guaranteed by Jorgensen's theory.) What happens to the Ford domain under further deformation? The answer is given by Fig. 0.6.

To describe it, we note that the "chain rule for isometric circles" (Lemmas 4.1.2 and 4.1.3) implies that the isometric circle $I(P_1^*)$ of $P_1^* := P_2P_1P_2$ passes through the vertex $P_2(v)$ of the upper polygon $P^+(\Gamma)$ in Fig. 0.5. Thus $Ih(P_1^*)$ is a hidden isometric hemisphere of $Ph(\Gamma)$. Moreover this isometric hemisphere and its translates by powers of K are the only hidden isometric hemispheres of $Ph(\Gamma)$ (Lemma 7.1.6). By using this fact we see that if we

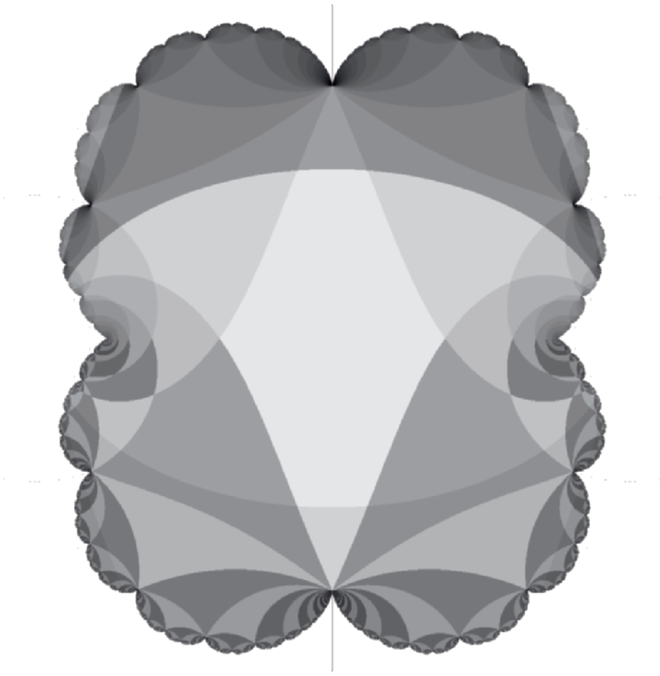

Fig. 0.13. $\mathcal{QF}$ in the slice of $\mathcal{X}$ at $a_0 = 1/3$

deform the group further then the Ford domain undergoes the following changes (see Fig. 0.6).

1. The hidden isometric hemisphere $Ih(P_1^*)$ breaks out through the vertex $P_2(v)$ and becomes to support a 2-dimensional face of $Ph(\Gamma)$.
2. The vertices v and $P_2(v)$ of the old upper polygon $P^+(\Gamma)$ lying in the complex place $\mathbb{C}$ are lifted to vertices of the new Ford domain $Ph(\Gamma)$ in the hyperbolic space $\mathbb{H}^3$.
3. The new upper polygon $P^+(\Gamma)$ is described by the sequence $\{P'_j\}$ defined by

$$P'_0 := P_0, \quad P'_1 := P_2, \quad P'_2 := P_1^* = P_2 P_1 P_2,$$
$$P'_{j+3n} := K^n P'_j K^{-n} \quad (j \in \{0,1,2\},\ n \in \mathbb{Z}).$$

Namely, the edges of $\partial P^+(\Gamma)$ are $I(P'_j) \cap P^+(\Gamma)$ $(j \in \mathbb{Z})$ in this order.

Moreover, the new Ford domain $Ph(\Gamma)$ is equal to the polyhedron $(\cap_j Eh(P_j)) \cap (\cap_j Eh(P'_j))$, and it is combinatorially dual to the *elliptic generator complex* in Fig. 0.14, which describes the relation between the two sequences $\{P_j\}$ and $\{P'_j\}$ (Definition 3.2.3).
The above description of the transition of the Ford domain is proved by using the following facts.

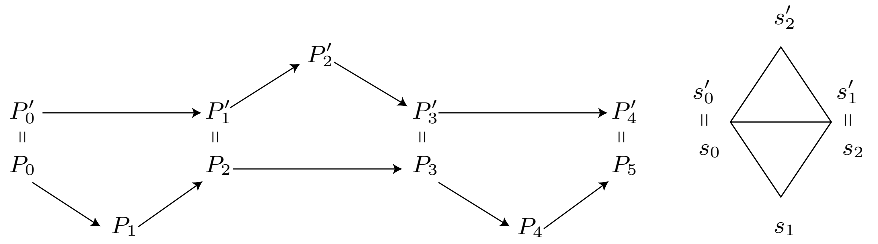

Fig. 0.14. Adjacent sequences of elliptic generators

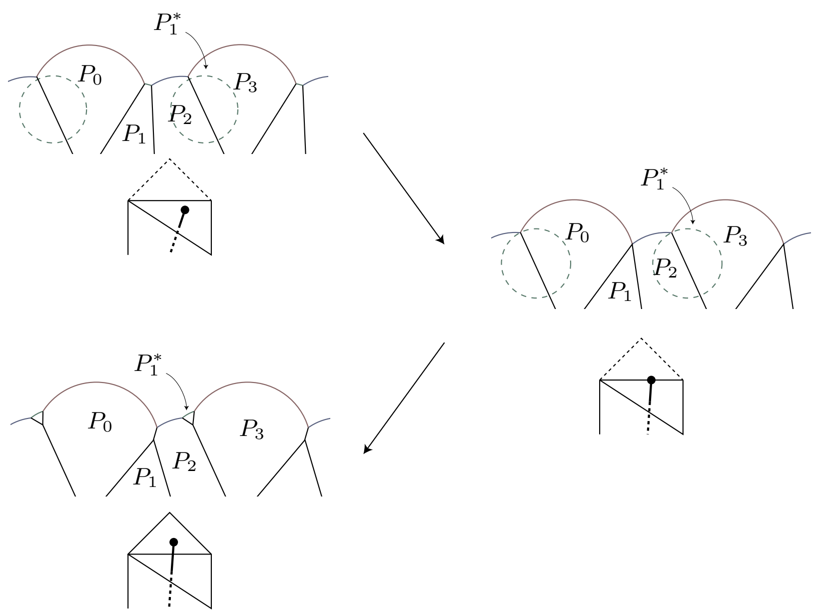

Fig. 0.15. Birth of a new face

1. The local behavior of the hidden isometric hemisphere $Ih(P_1^*)$ under small deformation is controlled by the "side parameter", which we explain in Subsection 5.7, and the only possible local behaviors are those illustrated in Fig. 0.15 (see Lemma 4.6.2).
2. The chain rule of the isometric circles (Lemma 4.1.3) guarantees that the above polyhedron $(\cap_j Eh(P_j)) \cap (\cap_j Eh(P_j'))$ satisfies the conditions of Poincare's theorem on fundamental polyhedra.

Jorgensen's theory shows that the transition described in the above is essentially the only possible transition of the combinatorial structure of the Ford domain when the group Γ is deformed in $\mathcal{QF}$ and that one can deform the group so that the Ford domain becomes arbitrary complicated (see Figs. 0.3–0.10).

5.6. Elliptic generators and the Farey triangulation

Observe that any successive triple $(P'_j, P'_{j+1}, P'_{j+2})$ in the bi-infinite sequence $\{P'_j\}$ introduced in the previous subsection is also an elliptic generator triple, i.e., it forms a generator system of Γ and $P'_{j+2}P'_{j+1}P'_j$ is equal to the parabolic transformation $K(z) = z + 1$. Such a bi-infinite sequence is called a *sequence of elliptic generators*, and an element of a sequence of elliptic generators is called an *elliptic generator* (cf. Definitions 2.1.1 and 2.1.13). By using the facts that the orbifold $\mathcal{O}$ is commensurable with the punctured torus T and that an elliptic generator corresponds to an essential simple loop in T, we can define the *slope* $s(P)$ for each elliptic generator P (Proposition 2.1.2 and Definition 2.1.3). Moreover, for any sequence $\{P_j\}$ of elliptic generators, the following hold.

1. $\{s(P_j)\}$ is a periodic sequence of period 3.
2. The set $\{s(P_0), s(P_1), s(P_2)\}$ spans a triangle of the Farey triangulation, or the modular diagram, $\mathcal{D}$ of the hyperbolic plane $\mathbb{H}^2$ (see Fig. 0.16).
3. $s(P'_2) = s(P_2P_1P_2)$ is the vertex opposite to the vertex $s(P_1)$ with respect to the edge $\langle s(P_0), s(P_2)\rangle = \langle s(P'_0), s(P'_1)\rangle$ in $\mathcal{D}$.

This gives a bijection between the sequences of elliptic generators (modulo shifts of indices by a multiple of 3) and the triangles of $\mathcal{D}$ (Proposition 2.1.10).

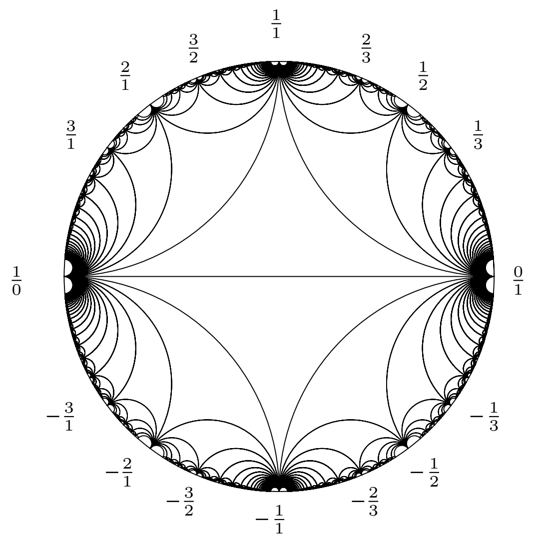

Fig. 0.16. Farey triangulation

Jorgensen's theory associates to each quasifuchsian punctured torus group $\Gamma \in \mathcal{QF}$ a pair (σ^-, σ^+) of triangles of $\mathcal{D}$, so that the combinatorial structure of the Ford domain $Ph(\Gamma)$ is completely determined by the pair (σ^-, σ^+)

(see Theorem 1.2.2). To be precise, (the boundaries of) the upper and lower polygons $P^+(\Gamma)$ and $P^-(\Gamma)$, respectively, are described by the sequences of elliptic generators associated with σ^+ and σ^-, and the Ford domain $Ph(\Gamma)$ is dual to the *elliptic generator complex* associated with the chain of triangles in $\mathcal{D}$ joining σ^- and σ^+ (Definition 3.2.3). If $\sigma^- = \sigma^+$, then the elliptic generator complex is identified with the real line $\mathbb{R}$ with vertex set $\mathbb{Z}$, where the vertex $j \in \mathbb{Z}$ corresponds to the j-th elliptic generator P_j. If σ^- and σ^+ are adjacent, then the elliptic generator complex is as illustrated in Fig. 0.14.

5.7. The side parameters of quasifuchsian punctured torus groups

The correspondence $\Gamma \mapsto (\sigma^-, \sigma^+)$ is refined to a (combinatorial) homeomorphism $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$, resembling the holomorphic isomorphism $\mathcal{QF} \cong \mathbb{H}^2 \times \mathbb{H}^2$ via the Bers' simultaneous uniformization theorem described in Subsection 5.4. To explain this homeomorphism, recall that the edges of the upper and lower polygons $P^+(\Gamma)$ and $P^-(\Gamma)$, respectively, are supported by the isometric circles of the sequences of elliptic generators $\{P_j^+\}$ and $\{P_j^-\}$ associated with σ^+ and σ^-. For $\epsilon \in \{-,+\}$ and $j \in \{0,1,2\}$, let θ_j^ϵ be the half of the angle of the circular edge of the polygon $P^\epsilon(\Gamma)$ contained in the isometric circle $I(P_j^\epsilon)$. Then the following identity holds (Proposition 4.2.16):

$$\theta_0^\epsilon + \theta_1^\epsilon + \theta_2^\epsilon = \frac{\pi}{2}.$$

Thus the triple $(\theta_0^\epsilon, \theta_1^\epsilon, \theta_2^\epsilon)$ can be regarded as $\pi/2$ times the barycentric coordinate of a point in the triangle $\sigma^\epsilon = \langle s(P_0^\epsilon), s(P_1^\epsilon), s(P_2^\epsilon)\rangle$ of (the abstract simplicial complex having the combinatorial structure of) the Farey triangulation $\mathcal{D}$. We denote the point by $\nu^\epsilon(\Gamma)$. Then $\nu^\epsilon(\Gamma)$ is not equal to a vertex of σ^ϵ and hence it is identified with a point in $\mathbb{H}^2 \cong |\mathcal{D}| - |\mathcal{D}^{(0)}|$, where $\mathcal{D}^{(0)}$ denotes the 0-skeleton of $\mathcal{D}$ and $|\cdot|$ denotes the underlying space of an abstract simplicial complex (cf. Sect. 1.3, p. 12). Set $\boldsymbol{\nu}(\Gamma) = (\nu^-(\Gamma), \nu^+(\Gamma))$ and call it the *side parameter* of Γ. Then Jorgensen's theory asserts that $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is a homeomorphism and that the combinatorial structure of the Ford domain $Ph(\Gamma)$ of $\Gamma \in \mathcal{QF}$ is completely described in terms of $\boldsymbol{\nu}(\Gamma)$ (see Main Theorem 1.3.5).

5.8. Jorgensen's theory for boundary groups.

A *marked punctured torus group* is (a representative of) an element of $\mathcal{X}$ which is discrete and isomorphic to the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. It is classically known that every marked group in the closure $\overline{\mathcal{QF}}$ of $\mathcal{QF}$ in $\mathcal{X}$ is a marked punctured torus group (see, for example, [55, Proposition 4.18]).

A group in $\overline{\mathcal{QF}} - \mathcal{QF}$ is called a *boundary group*. Moreover it is a consequence of Minsky's ending lamination theorem for punctured torus groups that every Kleinian punctured torus group is contained in the closure $\overline{\mathcal{QF}}$ (see [58]).

Jorgensen's theorem and its generalization enable us to get a visual understanding of these complicated groups (see Figs. 0.17, 0.19–0.21).

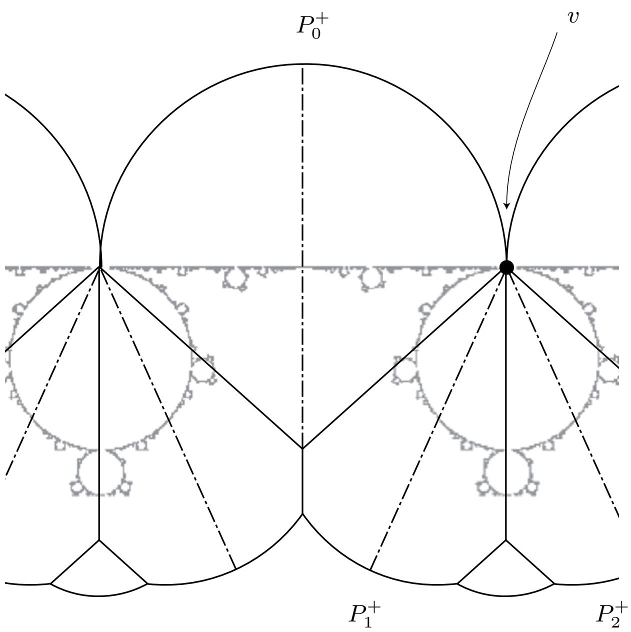

Fig. 0.17. $(\frac{1}{3}, \frac{1}{3} + \frac{\sqrt{5}}{6}\mathrm{i}, \frac{1}{3} - \frac{\sqrt{5}}{6}\mathrm{i})$ – Singly cusped group

A *singly cusped group*, Γ^*, is obtained from a quasifuchsian group Γ by a deformation which shrinks two successive edges of, say the upper polygon $P^+(\Gamma)$, into a single point, while fixing $\nu^-(\Gamma)$ (see Fig. 8 in Jorgensen [40] and Fig. 0.17). If the edges of $P^+(\Gamma)$ supported by the isometric circles $I(P_1^+)$ and $I(P_2^+)$ are shrunk into a point v, then the two fixed points of the loxodromic transformation KP_0^+ of Γ are united into the point v, and the corresponding element KP_0^+ of Γ^* becomes a parabolic transformation with parabolic fixed point v; this transformation is called an *accidental parabolic transformation*. The limit set of Γ^* is obtained from the Jordan curve $\Lambda(\Gamma)$ by pinching the two fixed points of each conjugate of (the old) KP_0^+ into a single point. Thus the "upper part" $\Omega^+(\Gamma^*)$ of the domain of discontinuity is not connected anymore; it consists of infinitely many (round) disks, whereas the lower component $\Omega^-(\Gamma^*)$ remains to be an open disk. Accordingly the quotient orbifold $\Omega^+(\Gamma^*)/\Gamma^*$ becomes an orbifold with underlying space a twice-punctured sphere with a cone point of cone angle π, or equivalently,

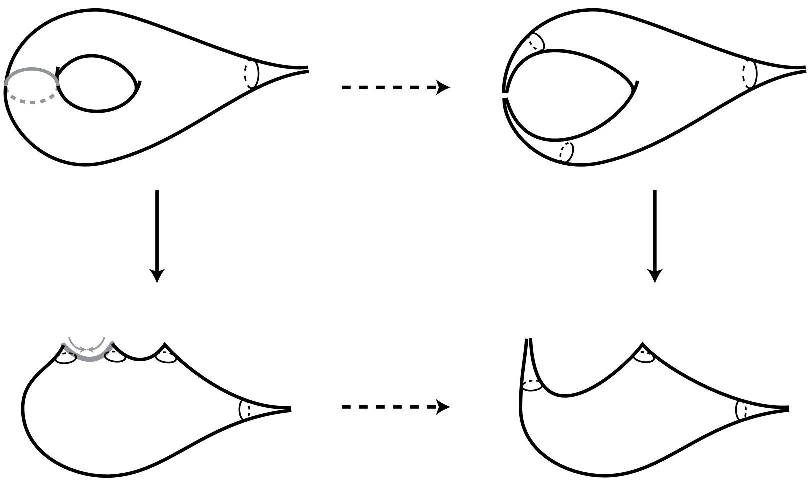

Fig. 0.18. Pinching the punctured torus $\Omega^+(\Gamma_0)/\Gamma_0$ and the quotient orbifold $\Omega^+(\Gamma)/\Gamma$

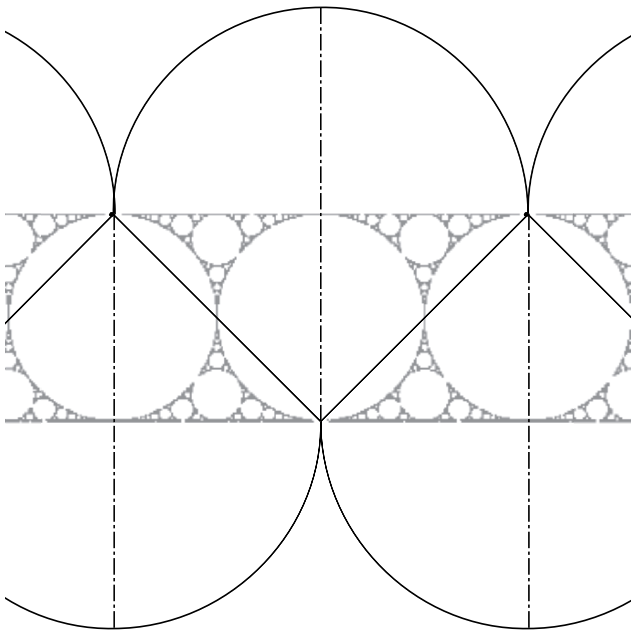

Fig. 0.19. $(\frac{1+\mathrm{i}}{2}, \frac{1-\mathrm{i}}{4}, \frac{1-\mathrm{i}}{4})$ – Doubly cusped group

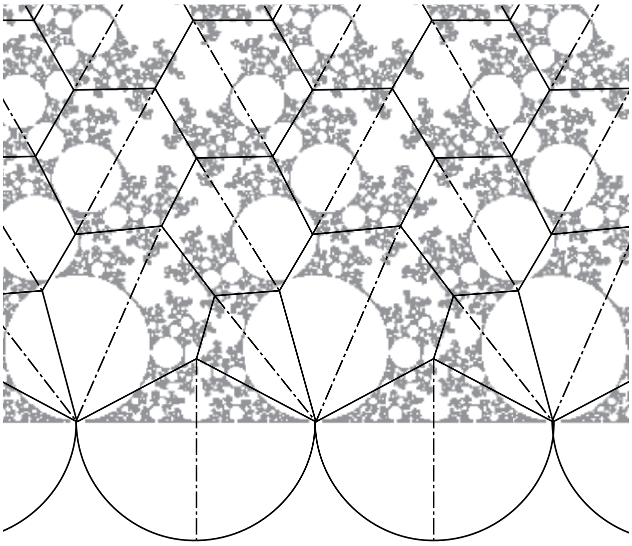

Fig. 0.20. (0.198264+0.376931i, 0.528473+0.142584i, 0.273263−0.519515i) – Singly degenerate group

the quotient of a triply punctured sphere which in turn is obtained from a once-punctured torus by pinching an essential simple loop corresponding to KP_0^+ to a point (see Fig. 0.18). The component $\nu^+(\Gamma)$ of the side parameter $\boldsymbol{\nu}(\Gamma)$ converges to the slope $s(P_0^+) \in \hat{\mathbb{Q}}$ of the elliptic generator P_0^+. The (generalized) Jorgensen's theory associates to the boundary group Γ^* the point $\boldsymbol{\nu}(\Gamma^*) := \lim \boldsymbol{\nu}(\Gamma)$ and gives a description of the Ford domain $Ph(\Gamma^*)$ in terms of $\boldsymbol{\nu}(\Gamma^*)$.

A *doubly cusped group*, Γ^*, is obtained from a quasifuchsian group Γ by a deformation which shrinks two successive edges of the upper polygon $P^+(\Gamma)$ into a single point and two successive edges of the lower polygon $P^-(\Gamma)$ into another single point (see Fig. 6 in Jorgensen [40] and Fig. 0.19). Then the limit set of Γ^* becomes a circle packing, and the corresponding side parameter $\boldsymbol{\nu}(\Gamma^*)$ is a pair of distinct rational numbers. For the simplest case where $\boldsymbol{\nu}(\Gamma^*)$ consists of a Farey neighbor, say $(0/1, 1/0)$, the limit set is the Apollonian packing, and as the parameter $\boldsymbol{\nu}(\Gamma^*)$ becomes complicated, to be precise, as the relative position between $\nu^-(\Gamma^*)$ and $\nu^+(\Gamma^*)$ becomes complicated, the limit set of the doubly cusped group becomes more and more intricate (see the beautiful Figs. 7.3, 9.15. 9.16 and 9.18 in [61]).

A singly or doubly cusped group remains to be *geometrically finite*, i.e., there is a finite volume submanifold of the quotient hyperbolic manifold which contains all closed geodesics, or equivalently, the quotient of the convex hull

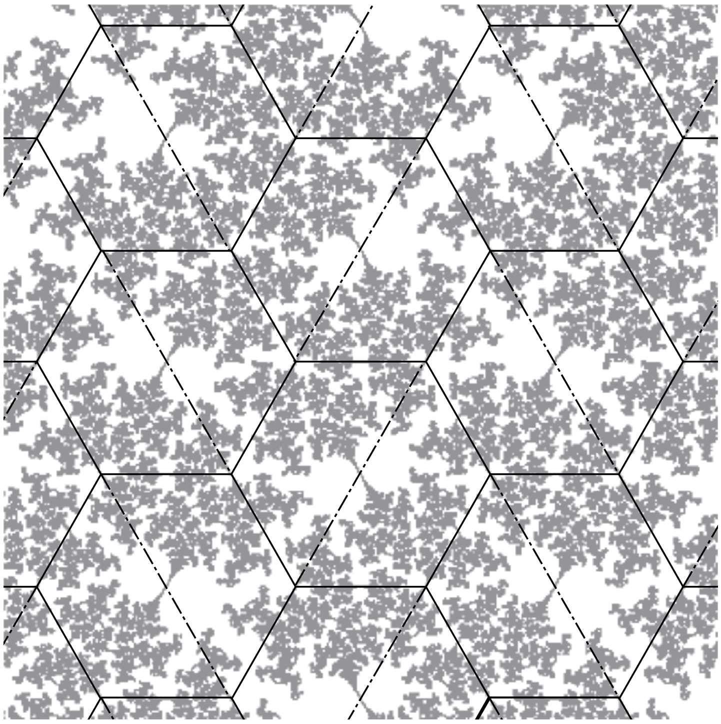

Fig. 0.21. $(\frac{1}{2}+\frac{1}{2\sqrt{3}}\mathrm{i}, -\frac{1}{\sqrt{3}}\mathrm{i}, \frac{1}{2}+\frac{1}{2\sqrt{3}}\mathrm{i})$ – Doubly degenerate group

of the limit set has finite volume. Constructions of more complicated *geometrically infinite* groups are presented below.

A *singly degenerate group*, Γ^*, is obtained as the limit of a sequence of quasifuchsian (or geometrically finite) groups $\{\Gamma_n\}$ such that $\nu^+(\Gamma_n)$ tends to an irrational boundary point, whereas $\nu^-(\Gamma_n)$ is fixed (see Fig. 0.20). Then the upper component $\Omega^+(\Gamma_n)$ of the domain of discontinuity becomes "smaller and smaller" as $n \to \infty$, and finally disappears at the limit. So, $\Omega(\Gamma^*)/\Gamma^*$ consists of only one component. A singly degenerate group is not geometrically finite anymore and hence is geometrically infinite.

A *doubly degenerate group* Γ^* is obtained as the limit of a sequence of quasifuchsian (or geometrically finite) groups $\{\Gamma_n\}$ such that $\boldsymbol{\nu}(\Gamma_n)$ tends to a pair of mutually distinct irrational boundary points (see Fig. 0.21). The limit set Γ^* is the whole Riemann sphere, and it gives rise to a sphere filling Peano curve (cf. [24], [13], [56], [18]). The special case, where $\boldsymbol{\nu}(\Gamma^*) := \lim \boldsymbol{\nu}(\Gamma_n)$ is the pair of the fixed points of the linear fractional transformation determined by a matrix $M \in SL(2,\mathbb{Z})$ with $|\operatorname{tr}(M)| \geq 3$, is particularly interesting to topologists. Because it gives the infinite cyclic cover of a hyperbolic punctured torus bundle over the circle with monodromy M. Figure 0.21 illustrates

the doubly degenerate group corresponding to the matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This gives the infinite cyclic cover of the figure-eight knot complement. Explicit constructions of this historically important group were given by Jorgensen and Marden [43] (cf. [40] and [41]) and Riley [67] independently and by completely different methods.

Recall that Minsky's ending lamination theorem [58] proves that the map assigning each $\Gamma \in \overline{\mathcal{QF}}$ with its end invariant gives a bijection from $\overline{\mathcal{QF}}$ to $\overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2}$ − diagonal$(\partial\mathbb{H}^2)$, extending the holomorphic isomorphism $\mathcal{QF} \cong \mathbb{H}^2 \times \mathbb{H}^2$. The first author proved, by using and resembling Minsky's theorem, that the combinatorial homeomorphism $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ extends to a surjective map $\boldsymbol{\nu} : \overline{\mathcal{QF}} \to \overline{\mathbb{H}^2} \times \overline{\mathbb{H}^2}$ − diagonal$(\partial\mathbb{H}^2)$ so that $\boldsymbol{\nu}(\Gamma)$ determines the combinatorial structure of the Ford domain for each $\Gamma \in \overline{\mathcal{QF}}$ (see [2]).

5.9. Extension of Jorgensen's theory beyond the quasifuchsian space.

Consider a doubly cusped group Γ with $\boldsymbol{\nu}(\Gamma) = (s^-, s^+)$, where s^- and s^+ are distinct vertices of $\mathcal{D}$. See Fig. 0.22 where $(s^-, s^+) = (\infty, 2/5)$. Assume, for simplicity, that s^- and s^+ are not Farey neighbors, i.e., the edge-path distance $d(s^-, s^+)$ is greater than 1. Let $\Sigma = (\sigma_1, \cdots, \sigma_m)$ be the chain of triangles of $\mathcal{D}$ intersecting the geodesic with endpoints s^- and s^+ in this order, and let $\Sigma^{(0)}$ be the set of the vertices of the triangles in Σ. Then, by (generalized) Jorgensen's theory, the Ford domain $Ph(\Gamma)$ is equal to the polyhedron

$$Eh(\Gamma, \Sigma) := \cap\{Eh(P) \mid P \text{ is an elliptic generator with } s(P) \in \Sigma^{(0)}\}.$$

For each $\epsilon \in \{-, +\}$, let P^ϵ be an elliptic generator with $s(P^\epsilon) = s^\epsilon$. Then the transformation $A^\epsilon := KP^\epsilon$ is an accidental parabolic transformation, whose parabolic fixed point is the point of tangency between $I(A^\epsilon) = I(P^\epsilon)$ and $I(KA^\epsilon K^{-1}) = K(I(P^\epsilon))$.

Now, perturb the group Γ in $\mathcal{X}$ so that each A^ϵ becomes an elliptic transformation (see Fig. 0.23). Note that its axis $\mathrm{Axis}\, A^\epsilon$ is equal to $Ih(A^\epsilon) \cap Ih(KA^\epsilon K^{-1})$ and its rotation angle θ^ϵ is equal to the exterior angle between $I(A^\epsilon)$ and $I(KA^\epsilon K^{-1})$. Generically, the resulting group Γ is not discrete anymore. However, we can see that the combinatorial structure of the corresponding polyhedron $Eh(\Gamma, \Sigma)$ is unchanged, except that a new edge contained in $\mathrm{Axis}\, A^\epsilon$ appears. Moreover the pairing transformations for the Ford domain of the original group continue to pair the faces of the new polyhedron $Eh(\Gamma, \Sigma)$. (To be precise, face pairings are defined for the quotient of $Eh(\Gamma, \Sigma)$ by $\langle K \rangle$.) The space obtained from $Eh(\Gamma, \Sigma)$ by pairing the faces turns out to be a hyperbolic cone manifold commensurable with the hyperbolic cone manifold $M(2\theta^-, 2\theta^+)$ in Sect. 3. We can show that this remains valid as long as $0 \le \theta^\epsilon < \pi$ (see Fig. 0.24).

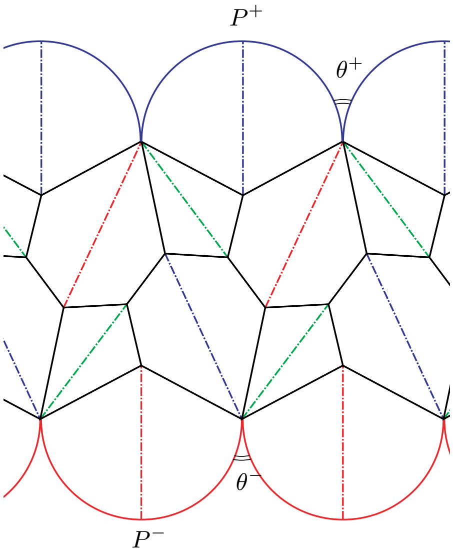

Fig. 0.22. $(0.525833 + 0.110676\mathrm{i}, 0.207579 + 0.389324\mathrm{i}, 0.266588 - 0.5\mathrm{i})$

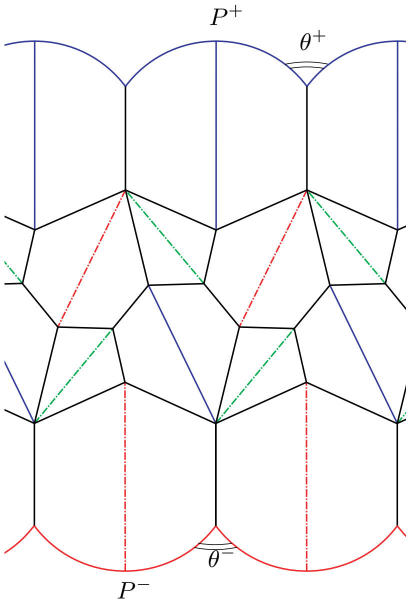

Fig. 0.23. $(0.483147 + 0.115832\mathrm{i}, 0.232288 + 0.514691\mathrm{i}, 0.284565 - 0.630523\mathrm{i})$

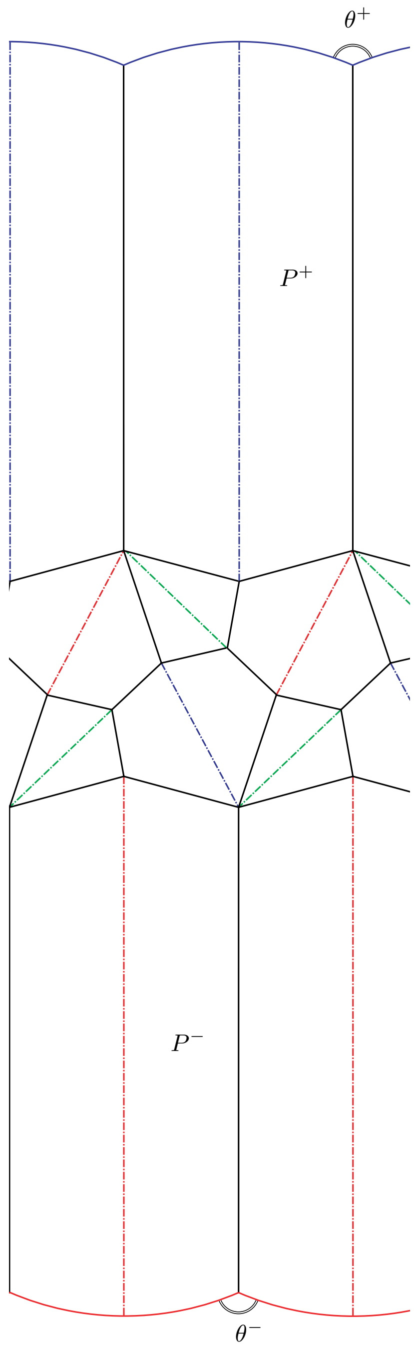

Fig. 0.24. $(0.348619 + 0.115197\mathrm{i}, 0.310165 + 1.1507\mathrm{i}, 0.341216 - 1.265900\mathrm{i})$

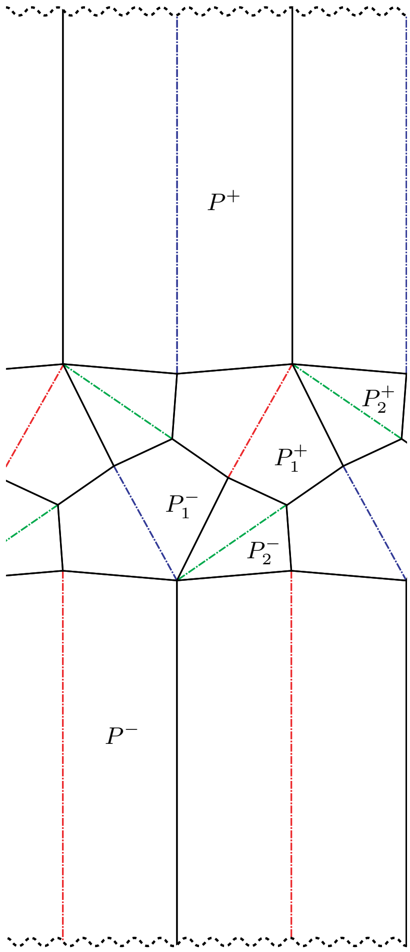

Fig. 0.25. $(0.136324 + 0.064967\mathrm{i}, 0.429022 + 4.94894\mathrm{i}, 0.434654 - 5.0139100\mathrm{i})$

As the angle θ^ϵ approaches π, the radius of $Ih(A^\epsilon)$ increases to ∞ and $Ih(A^\epsilon)$ converges to the vertical plane above a line in $\mathbb{C}$ parallel to the real line (see Fig. 0.25). So $Eh(P^\epsilon) = Eh(A^\epsilon)$ converges to one of the two half spaces bounded by the limit vertical plane. When θ^ϵ becomes π, the transformations A^ϵ and P^ϵ become π-rotations around vertical geodesics. Though the isometric hemisphere of P^ϵ is not defined, we continue to denote the above limit half space by $Eh(P^\epsilon)$.

We now explain what happens to the corresponding $Eh(\Gamma, \Sigma)$ when (θ^-, θ^+) becomes (π, π).

Suppose $d(s^-, s^+) \geq 3$, and assume $(s^-, s^+) = (1/0, q/p)$ for simplicity. Then $Eh(\Gamma, \Sigma)$ continues to be a 3-dimensional polyhedron and the pairing transformations for the Ford domain of the original group continue to pair

the faces of $Eh(\Gamma, \Sigma)$. The space obtained from $Eh(\Gamma, \Sigma)$ by pairing the faces turns out to be a hyperbolic orbifold commensurable with the complement of the 2-bridge link of type (p, q). The only change in the combinatorial structure of $Eh(\Gamma, \Sigma)$, except the above change in $Eh(P^\epsilon)$, is that the faces P_1^+ and P_2^+ (resp. P_1^- and P_2^-) in Fig. 0.25 are united into the single face Q^+ (resp. Q^-) in Fig. 0.26. The actual Ford domain of the group Γ is equal to the union of the images of $Eh(\Gamma, \Sigma)$ by the infinite dihedral group $\langle P^-, P^+\rangle$, which acts on $\mathbb{H}^3$ as Euclidean isometries (see Fig. 0.26).

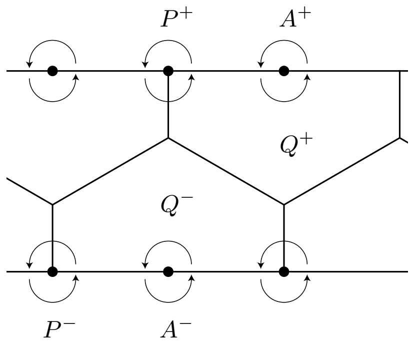

Fig. 0.26.

Suppose $d(s^-, s^+) = 2$, and assume $(s^-, s^+) = (1/0, 1/p)$ with $p \geq 3$ for simplicity. Then, as (θ^-, θ^+) approaches (π, π), $Eh(P^-) \cap Eh(P^+)$ degenerates into (a subset of) a vertical plane, and $Eh(\Gamma, \Sigma)$ degenerates into a 2-dimensional polyhedron contained in the vertical plane. See Figs. 0.27–0.29 where $p = 4$. The limit group Γ preserves the vertical plane, and the polygon $Eh(\Gamma, \Sigma)$ in the vertical plane is regarded as the Ford polygon of the action of Γ on the vertical plane. Moreover Γ is commensurable with the orbifold fundamental group of the base orbifold of the Seifert fibered structure of the complement of the 2-bridge link of type $(p, 1)$.

6. Reformulation of Jorgensen's theory for quasifuchsian punctured torus groups

For convenience, we regard a marked group Γ representing an element of $\mathcal{X}$ as the image of a *type-preserving representation* $\rho : \pi_1(\mathcal{O}) \to PSL(2, \mathbb{C})$, and identify the space $\mathcal{X}$ with the space of equivalence classes of type-preserving representations (Definitions 2.2.1 and 2.2.6). Here $\pi_1(\mathcal{O})$ is the orbifold fundamental group of the orbifold $\mathcal{O}$ and is isomorphic to the free product $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. The fundamental group $\pi_1(T)$ of the once-punctured torus T is an index 2 subgroup of $\pi_1(\mathcal{O})$, and there is a one-to-one

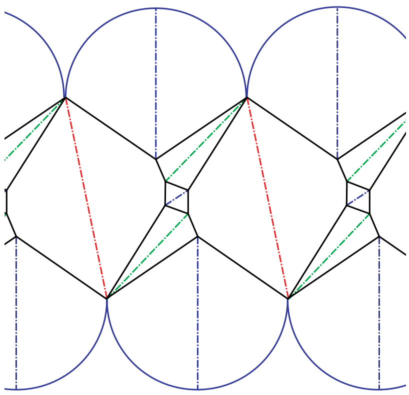

Fig. 0.27. $(0.404256 - 0.254426i, 0.209822 - 0.303145i, 0.385922 + 0.557571i)$ – Doubly cusped group

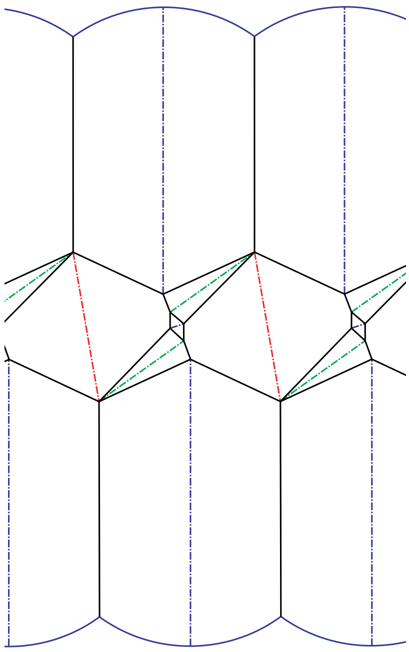

Fig. 0.28. $(0.273239 - 0.269885i, 0.30051 - 0.644994i, 0.426251 + 0.914879i)$

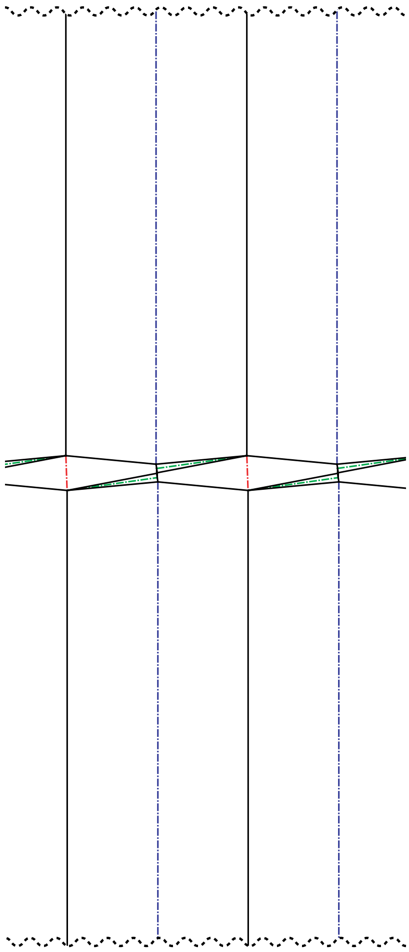

Fig. 0.29. $(0.0186119 - 0.0948854\mathrm{i}, 0.486063 - 4.97939\mathrm{i}, 0.495325 + 5.07428\mathrm{i})$

correspondence between type-preserving representations of $\pi_1(T)$ and those of $\pi_1(\mathcal{O})$ (Proposition 2.2.2).

Then a marked quasifuchsian punctured torus group Γ is identified with a quasifuchsian representation ρ, and we denote the Ford domain $Ph(\Gamma)$ and the side parameter $\boldsymbol{\nu}(\Gamma)$, respectively, by $Ph(\rho)$ and $\boldsymbol{\nu}(\rho)$. Jorgensen's theory says that $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$ is a homeomorphism and $\boldsymbol{\nu}(\rho)$ encodes the combinatorial structure of the Ford domain $Ph(\rho)$. By abuse of notation, we often denote the image $\boldsymbol{\nu}(\rho) = (\nu^-(\rho), \nu^+(\rho)) \in \mathbb{H}^2 \times \mathbb{H}^2$ by the symbol $\boldsymbol{\nu} = (\nu^-, \nu^+)$. Under this notation, let $\Sigma(\boldsymbol{\nu})$ be the chain of triangles of the Farey triangulation intersecting the geodesic segment $[\nu^-, \nu^+]$ joining ν^- with ν^+ (Definition 3.3.3). Then Jorgensen's theory says that the combinatorial structure of the Ford domain $Ph(\rho)$ is dual to a certain abstract simplicial

complex $\mathcal{L}(\boldsymbol{\nu})$, called the *elliptic generator complex* associated with $\boldsymbol{\nu}$, which is constructed from $\Sigma(\boldsymbol{\nu})$ in Sect. 3.3 (p. 44). In particular, the faces of the Ford domain are supported by the isometric hemispheres of the images by ρ of the elliptic generators of $\pi_1(\mathcal{O})$ whose slopes are vertices of $\Sigma(\boldsymbol{\nu})$ (cf. Definitions 2.1.1 and 2.1.3).

By abuse of notation again, we often denote a point in $\mathbb{H}^2 \times \mathbb{H}^2$ by the symbol $\boldsymbol{\nu} = (\nu^-, \nu^+)$, and we call it a *label*. Then we are led to the concept of a *labeled representation*, which is defined to be a pair $\boldsymbol{\rho} = (\rho, \boldsymbol{\nu})$ of a type-preserving representation $\rho : \pi_1(\mathcal{O}) \to PSL(2, \mathbb{C})$ and a label $\boldsymbol{\nu} = (\nu^-, \nu^+) \in \mathbb{H}^2 \times \mathbb{H}^2$ (Definition 3.3.2). We say that $\boldsymbol{\rho}$ is *quasifuchsian* if (i) ρ is quasifuchsian, (ii) the Ford domain $Ph(\rho)$ is as described by Jorgensen, and (iii) $\boldsymbol{\nu}$ is equal to the side parameter of ρ (Definition 6.1.1). Our first task is to give a characterization of quasifuchsian labeled representations, which is suitable for the proof of Main Theorem 1.3.5. This is done by Theorem 6.1.8 (Good implies quasifuchsian), which shows that if a labeled representation is "good", then it is quasifuchsian. Here a labeled representation is defined to be *good* if it satisfies the three conditions, Nonzero, Frontier and Duality (Definition 6.1.7). Though these conditions are rather complicated, it is not difficult to check if a given labeled representation satisfies them. Thus Theorem 6.1.8 gives us a practical method for checking if a given labeled representation is quasifuchsian. (Actually, Modified Main Theorem 6.1.11 implies that the converse to Theorem 6.1.8 is also valid, and hence a labeled representation is quasifuchsian if and only if it is good.) The proof of Theorem 6.1.8 is done by using Poincare's theorem on fundamental polyhedra in Sects. 6.3 (p. 138), 6.4 (p. 142) and 6.5 (p. 144).

Let $\mathcal{J}[\mathcal{QF}] \subset \mathcal{QF} \times (\mathbb{H}^2 \times \mathbb{H}^2)$ be the space of all good labeled representations. Then Jorgensen's result is equivalent to Modified Main Theorem 6.1.11 which asserts that the projections $\boldsymbol{\mu}_1 : \mathcal{J}[\mathcal{QF}] \to \mathcal{QF}$ and $\boldsymbol{\mu}_2 : \mathcal{J}[\mathcal{QF}] \to \mathbb{H}^2 \times \mathbb{H}^2$ are homeomorphisms.

7. Idea of the proof that $\boldsymbol{\mu}_1$ is a homeomorphism

The injectivity of $\boldsymbol{\mu}_1$ is guaranteed by Proposition 6.2.5 (Uniqueness of good label), which is an easy consequence of the definition and is proved in Sect. 6.4 (p. 142). So the main task is the proof of surjectivity. Namely, we need to show that the combinatorial structure of the Ford domain of a given quasifuchsian representation is as described by Jorgensen. Roughly speaking, the proof amounts to showing that under a continuous deformation of the quasifuchsian representations, combinatorial transitions of the Ford domain happen only at their frontier in the ideal boundary, i.e., in the complex plane $\mathbb{C}$, as illustrated in Fig. 4.15, and the combinatorial structure in the hyperbolic space is stable. The actual proof consists of the following two steps.

1. Proposition 6.2.1 (Openness), which guarantees the openness of the image of the projection $\boldsymbol{\mu}_1 : \mathcal{J}[\mathcal{QF}] \to \mathcal{QF}$ in $\mathcal{QF}$

2. Propositions 6.2.2 (SameStratum) and 6.2.4 (Closedness), which guarantee the closedness of the image of the projection $\boldsymbol{\mu}_1 : \mathcal{J}[\mathcal{QF}] \to \mathcal{QF}$ in $\mathcal{QF}$

Since we can easily see that the image of $\boldsymbol{\mu}_1$ is non-empty, i.e., there exists a good labeled representation (Proposition 5.1.3), and since $\mathcal{QF}$ is connected, the above results imply that $\boldsymbol{\mu}_1 : \mathcal{J}[\mathcal{QF}] \to \mathcal{QF}$ is surjective.

Step 1 is completed in Chap. 7, and a rough idea for this step is as follows. To show the openness of the image $\boldsymbol{\mu}_1(\mathcal{J}[\mathcal{QF}])$ in $\mathcal{QF}$ around $\rho = \boldsymbol{\mu}_1(\rho, \boldsymbol{\nu})$, we pick up a set of elliptic generators of $\pi_1(\mathcal{O})$, which is finite modulo conjugation by the peripheral element, and study how the pattern of the corresponding isometric hemispheres change after a small perturbation of the representation ρ. By virtue of the key Lemmas 4.6.1, 4.6.2 and 4.6.7 (see Fig. 4.15) and Lemma 7.2.3 (disjointness), we see that each nearby representation ρ' has a label $\boldsymbol{\nu}'$ such that $(\rho', \boldsymbol{\nu}')$ is good. We thus obtain the openness of $\boldsymbol{\mu}_1(\mathcal{J}[\mathcal{QF}])$. We note that Schedler [72] obtains the corresponding result, by establishing a general stability theorem of the Ford domains using the theory of holomorphic motions. However, we do not know if the idea of holomorphic motions can be applied in the proof of the generalization of Jorgensen's theory to hyperbolic cone-manifolds.

Step 2 is completed in Chap. 8. To describe a rough idea, let $\{(\rho_n, \boldsymbol{\nu}_n)\}$ be a sequence of good labeled representations such that $\lim \rho_n \in \mathcal{QF}$. Then Proposition 6.2.2 (SameStratum) guarantees that some subsequence satisfies the condition SameStratum (Definition 6.2.2). Roughly speaking, this conditions says that the Ford domains $Ph(\rho_n)$ have the same combinatorial structure and therefore we can talk about the "behavior of a face (or an edge or a vertex) of $Ph(\rho_n)$ as $n \to \infty$". The proof of Proposition 6.2.2 is done in Sect. 8.1 (p. 172), and it is based on the fact that the convergence $\rho_n \to \rho_\infty$ is strong and a certain lemma (Lemma 8.1.1) due to Jorgensen, which is a prototype of Minsky's pivot theorem in [58].

Proposition 6.2.4 (Closedness) is proved in Sects. 8.3 (p. 180) - 8.12 (p. 209) in Chap. 8, and its main ingredient is to show that no unexpected degeneration of a face of $Ph(\rho_n)$ happens as $n \to \infty$. This is the most involved part of this monograph. A reason why it is so involved is that we have to list all possible degenerations, before showing degenerations do not happen. However, as is found in Jorgensen's original argument [40], the idea to prohibit degenerations consists of only a few simple observations (see the introduction to Chap. 8). Another reason for the complication in this step (and the previous step) lies in the treatment of the 'thin' case, i.e., the case when both components of $\boldsymbol{\nu}_\infty = \lim \boldsymbol{\nu}_n$ belong to the interior of a single edge τ of the Farey triangulation. However, we can treat this special case by using the results established in Sect. 5.2 (p. 106).

8. Idea of the proof that $\boldsymbol{\mu}_2$ is a homeomorphism

The proof consists of the following two steps.

1. Proposition 6.2.7 (Convergence), which shows that, for a sequence of good labeled representations $\{(\rho_n, \boldsymbol{\nu}_n)\}$, if $\lim \boldsymbol{\nu}_n \in \mathbb{H}^2 \times \mathbb{H}^2$ exists (and if it satisfies the condition SameStratum), then it has a subsequence such that the corresponding subsequence of $\{\rho_n\}$ converges in $\overline{\mathcal{QF}}$.
2. Chapter 9, where the proof of bijectivity of the map $\boldsymbol{\mu}_2$ is established by using Propositions 6.2.2 (SameStratum), 6.2.4 (Closedness) and 6.2.7 (Convergence) and an elementary intersection theory in algebraic geometry.

We describe a rough idea of Step 2. In Lemma 4.2.18, we see that if $(\rho, \boldsymbol{\nu})$ is a good labeled representation which belongs to the inverse image of a label $\boldsymbol{\nu} \in \mathbb{H}^2 \times \mathbb{H}^2$ by $\boldsymbol{\mu}_2$, then ρ (to be precise, a *Markoff map* inducing ρ) satisfies a certain algebraic equation. What we need to do is to single out a unique 'geometric' root among numerous roots of an algebraic equation associated with $\boldsymbol{\nu}$. To this end, we introduce the concept of the *geometric multiplicity* $d_G(\boldsymbol{\nu})$ for each $\boldsymbol{\nu} = (\nu^-, \nu^+)$ by using the elementary intersection theory in algebraic geometry (Definition 9.2.1). Then the bijectivity of the projection $\boldsymbol{\mu}_2 : \mathcal{J}[\mathcal{QF}] \to \mathbb{H}^2 \times \mathbb{H}^2$ is equivalent to the assertion that $d_G(\boldsymbol{\nu}) = 1$ for every label $\boldsymbol{\nu}$. After making a detailed study of the algebraic varieties associated with the labels (Sect. 9.1, p. 215), we show in Corollary 9.2.7 that $d_G(\boldsymbol{\nu})$ does not depend on $\boldsymbol{\nu}$ by using the "continuity of roots" (Lemmas 9.3.1 and 9.3.3) and Propositions 6.2.2 (SameStratum), 6.2.4 (Closedness) and 6.2.7 (Convergence). On the other hand, it is easy to see that $d_G(\boldsymbol{\nu}) = 1$ if $\boldsymbol{\nu}$ belongs to the diagonal set, i.e., $\nu^- = \nu^+$ (Proposition 5.1.5). We thus obtain the bijectivity of $\boldsymbol{\mu}_2 : \mathcal{J}[\mathcal{QF}] \to \mathbb{H}^2 \times \mathbb{H}^2$, and hence that of $\boldsymbol{\nu} : \mathcal{QF} \to \mathbb{H}^2 \times \mathbb{H}^2$. We note that, though the bijectivity is claimed in [40], no indication of the proof is presented. The arguments outlined here is new in this sense. Actually, we were able to complete this step only fairly recently.

9. Organization of the monograph

The monograph consists of nine chapters and an appendix.

In Chap. 1, we reformulate Jorgensen's theory from the 3-dimensional viewpoint, and give a conceptual description of Jorgensen's theorem (Theorems 1.2.2, 1.3.2 and Main Theorem 1.3.5). We also give a description of the ideal tetrahedral complex which is a geometric dual to the Ford domain (Theorem 1.4.2). This chapter is essentially equal to the announcement in [10].

In Chap. 2, we describe the intimate relations among the Fricke surfaces, namely the punctured torus T, the four-times punctured sphere S, and the $(2, 2, 2, \infty)$-orbifold $\mathcal{O}$. We first give a complete description of the 'geometric'

generator systems of the fundamental groups of the Fricke surfaces (Propositions 2.1.6 and 2.1.9) and describe their relation with the Farey triangulation (Proposition 2.1.10). Then we show the equivalence among the spaces of the type-preserving representations (Definition 2.2.1) of the fundamental groups of Fricke surfaces (Proposition 2.2.2). The concepts of the Markoff maps and complex probabilities are introduced in Sects. 2.3 and 2.4, respectively, and explicit matrices for the type-preserving representations and its intuitive description are given (Lemma 2.3.7 and Proposition 2.4.4). Though almost all contents in this chapter seem to be known to the experts, we could not find explicit reference for some of the contents. Thus we included full proofs for all propositions in this chapter.

In Chap. 3, we introduce the definitions of a labeled representation $\boldsymbol{\rho} = (\rho, \boldsymbol{\nu})$, the complex $\mathcal{L}(\boldsymbol{\nu})$, which is a combinatorial dual to the Ford domain, and the virtual Ford domain $Eh(\boldsymbol{\rho})$. These are used in Chap. 6 to reformulate Main Theorem 1.3.5.

In Chap. 4, we first give a detailed proof to the chain rule for isometric circles (Lemma 4.1.2), on which the whole argument is based. We then introduce Jorgensen's side parameter (Definition 4.2.9), and prove various properties of the side parameter.

In Chap. 5, we give a detailed study of some special examples. In particular, complete descriptions of real representations and *isosceles* representations are given in Sects. 5.1 and 5.2. Though isosceles representations themselves are simple objects, their neighborhood in $\mathcal{QF}$ is rich in variety. However, they are essentially controlled by side parameters (Proposition 5.2.13). These representations also form the starting point toward the proof of Jorgensen's theory. In Sect. 5.3, we describe how two-parabolic groups arise as the images of type-preserving representations of certain kind, and explain the reason why the generalized Jorgensen's theory is useful to the study of the 2-bridge links.

In Chap. 6, we give a 2-dimensional reformulation of Main Theorem 1.3.5 (Modified Main Theorem 6.1.11). This is more similar to Jorgensen's original description than the three dimensional picture described in Chap. 1, and is rather complicated. However, it fits with the proof given in this monograph. The equivalence between the two formulations is guaranteed by Theorem 6.1.8, which is proved by using (a variation of) Poincare's theorem on fundamental polyhedron. In Sect. 6.2, we give a route map for the proof of Modified Main Theorem 6.1.11. At the end of this chapter, we prove certain properties of the elements in $\mathcal{QF}$ which are useful in the actual computation of the Ford domains (see Propositions 6.7.1 and 6.7.2). These properties are also used in the Bers' slice project in [50] (cf. [82]).

The remaining Chaps. 7, 8 and 9 are devoted to the proof of Modified Main Theorem 6.1.11. To be precise, Steps 1 and 2 , respectively, presented in Sect. 7 are completed in Chaps. 7 and 8, and Steps 1 and 2 , respectively, presented in Sect. 8 are completed in Chaps. 8 and 9. (See also Sect. 6.2 for the route map.)

In the Appendix, we give a proof to some of the basic facts concerning the Ford domain.

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We would like to express our deepest thanks to Troels Jorgensen for his explanations of his results, invaluable suggestions, and for the beautiful mathematics he had produced. We were very much encouraged when he told us he had got an inspiration that there should be some relationship between his work and 2-bridge links. We also thank his collaborator, Al Marden, for his interest on our work and encouragement. It was great pleasure to explain our approach to him and Caroline Series at University of Warwick. We sincerely thank Caroline Series for her kind hospitality at University of Warwick. Comparison of our work and her joint works with Linda Keen on punctured torus groups lead us to the related works [7, 10]. We were stimulated very much by attending the wonderful workshops organized by her. We also thank members and visitors of University of Warwick, including David Epstein, Brian Bowditch, Young Eun Choi, Raquel Diaz and Mark Lackenby for their interest and encouragement. We thank Brian Bowditch for sending us his mysterious paper [17] and for fruitful discussion, which brought the first and second authors to the related works with Hideki Miyachi [5, 6]. We thank Travis Schedler for sending us his beautiful paper [72] on Jorgensen's work, which was partially influenced by our preliminary announcement [9]. Interplay with his paper has improved the monograph very much. In particular, Chap. 7 and Appendix A.1 were much influenced by his paper. We thank Norbert A'Campo, Michael Heusener, Eriko Hironaka and Kazuhiro Konno for helpful discussion concerning Chap. 9. We thank Stephan Hamperies, Heinz Helling, Richard Weidmann and Heiner Zieschang for their helpful suggestions concerning Sect. 2.1. Stephan Hamperies taught us a neat proof of Proposition 2.1.6, which greatly simplifies our original proof. We thank Alexander Mednykh for discussion concerning Proposition 2.2.2. We thank Norbert A'Campo, Gerhard Burde, Shungbok Hong, Yoichi Imayoshi and Masaaki Yoshida for inviting the second author to series of lectures on our work at University Basel, Goethe University Frankfurt am Main, Pohang University, Osaka City University and Kyushu University. Yohei Komori encouraged us to organize a workshop to explain the first version of the monograph. The workshop was held at Osaka University during 12-15, September, 2005. We thank the participants of the workshop, in particular to Michihiko Fujii, Yohei Komori, Hideki Miyachi, Toshihiro Nakanishi, Yoshihide Okumura, Ken'ichi Ohshika, Hiroki Sato, Kenneth Shackleton, Masahiko Taniguchi, Akira Ushijima and Xiantao Wang for their devotion, patience and suggestions. We thank Kenneth Shackleton and Yoshihide Okumura for their careful reading of the first version. We also thank Colin Adams, Iain Aitchison, Michel Boileau, Warren Dicks, Craig Hodgson, Sadayoshi Kojima, John Parker, Joan Porti, Alan Reid, Toshiyuki

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