# Part III

# Application to the spherical mean operator

We apply the concept of the distributional approximate inverse, outlined in Chapter 4, to the problem of recovering a function from its spherical means. The elucidations in this part mainly are subject of the article [115][1] by Schuster and Quinto. We focus at the inversion of the *spherical mean operator*

$$\mathbf{M}f(a_0,r) = \int_{S(a_0,r)} f(x)\,\mathrm{d}S_n^r(x)\,, \quad S(a_0,r) = \{x \in \mathbb{R}^n : \|x-a_0\| = r\}\,,$$

$a_0 \in A \subset \mathbb{R}^n$. This problem arises in a variety of applications such as thermoacoustic detection of tumours, see Kruger et al. [60], Xu, Wang [133], seismics, see Romanov [104], and SONAR (SOund in NAvigation and Radiation), see Lavrentiev et al. [62], Louis, Quinto [74]. A detailed outline of the relation between spherical means and the detection of reflectivity of the earth's surface by SAR (Synthetic Aperture Radar) can be found in Cheney [15].

This is reason enough to develop inversion schemes for $\mathbf{M}$, where the different models differ from each other by the *center sets* $A$. Norton [88] gives an inversion formula for the case where the center set is a circle in a plane, whereas Finch et al. [31] consider the situation where $A$ is the boundary of a bounded, connected and open set in $\mathbb{R}^n$. In Ramm [96] a proof of injectivity of $\mathbf{M}$ for the latter case can be found. Denisjuk [19] considers the general case, where the centers are located on hyperplanes in $\mathbb{R}^n$. He gives an inversion method, which relies on a transform mapping spheres to planes and algebraic reconstruction techniques (ART). The problem of limited data is also treated in this article. Andersson [5] investigates the properties of $\mathbf{M}$ as a bounded mapping between distribution spaces, where the center set is the hyperplane $x_{n+1} = 0$ in $\mathbb{R}^{n+1}$. He deduces an inversion formula with an analogue structure as that of the Radon transform. The computation of reconstruction kernels outlined in Chapter 13 relies on this very formula.

A microlocal analysis for $\mathbf{M}$ has been developed in Louis, Quinto [74] to clarify which singularities of the object to be recovered can be visualized from the given data and which cannot. They prove that only those singularities of $f$ being conormal to the sphere $S(a_0,r)$ can be detected.

This part is organized as follows. The first chapter is dedicated to the investigation of the spherical mean operator $\mathbf{M}$. We shortly summarize the mathematical models in SONAR and SAR (Section 11.1) and the mathematical properties of $\mathbf{M}$ which are proved in Andersson [5] (Section 11.2). Just as in [5] we only consider the case where the center set $A$ is a special hyperplane.

[1] 'Some pictures in Part III are taken from T. Schuster and E.T. Quinto, *On a regularization scheme for linear operators in distribution spaces with an application to the spherical Radon transform*, SIAM J. Appl. Math., 65 (2005), pp. 1369–1387. Copyright ©2005 Society for Industrial and Applied Mathematics. Reprinted with permission.'

Chapter 11 concludes with the formulation of the method of approximate inverse and its semi-discrete version which is important from a practical point of view. The following chapter deals with the design of an operator adjusted mollifier. The computation of corresponding reconstruction kernels follows in Chapter 13. Chapter 14 shows how the method works with the help of numerical results.