Part II

Application to 3D Doppler Tomography

Vector and tensor tomography is a relatively new kind of tomography compared to conventional X-ray CT. Vector field tomography is the reconstruction of the velocity or vorticity of a moving fluid or gas from a set of integrals. These integrals are obtained e.g. by time-of-flight measurements or applying an electric field. When the measurement procedure uses ultrasound and relies on the *Doppler effect*, then we call this particular vector tomography problem *Doppler tomography*. Applications of vector and tensor tomography include medical diagnosis, oceanography, plasma physics or photoelasticity to name only a few. We refer to SPARR AND STRÅHLÉN [119] for a comprehensive overview.

The measured data often are integrals along lines over projections of the vector field:

$$y(\mathbf{f}, L) = \int_L \langle \theta, \mathbf{f}(x) \rangle \, \mathrm{d}\ell(x) \,.$$

Here θ is a unit vector. In Doppler tomography $\theta = \theta_L$ is the vector of direction corresponding to the line L. The mapping $\mathbf{f} \mapsto y(\mathbf{f}, \cdot) =: \mathbf{Df}$ is then called *Doppler transform*. It is equal to the first moment of the velocity spectrum and coincides with the longitudinal ray transform for vector fields in SHARAFUTDINOV [116]. JUHLIN [50] outlined how to get the Doppler transform of a searched for velocity field $\mathbf{f}$ as data with the help of ultrasound Doppler measurements. In JANSSON ET AL. [47] the authors developed an experimental setup at the Lund Institute of Technology (Sweden). The inverse problem thus consists of recovering the flow $\mathbf{f}$ from its Doppler transform. In SPARR ET AL. [120], SHARAFUTDINOV [116] and [109] the authors investigate the mathematical properties of $\mathbf{D}$. As an important result they state that only the *solenoidal part* of $\mathbf{f}$ can be detected from $\mathbf{Df}$ since the Doppler transform has a non-trivial null space consisting of potential fields and hence is not injective. This is an essential difference from the Radon transform.

In recent years various methods to recover vector and tensor fields from integral measurements have been established. In NORTON [88], BRAUN AND HAUCK [12] and SHARAFUTDINOV [116], the authors prove inversion formulas which yield the solenoidal part of $\mathbf{f}$. WINTERS, ROUSEFF [131] present an algorithm of filtered backprojection type to compute the curl of $\mathbf{f}$. In OSMAN, PRINCE [90] a vector field tomography problem in three dimensions on bounded domains with arbitrary boundary conditions is considered. DESBAT [24] investigates optimal sampling schemes by transferring ideas from scalar computerized tomography. An iterative scheme for inverting $\mathbf{D}$ based on algebraic reconstruction techniques is outlined in WERNSDÖRFER [129]. Least squares methods are considered in BEZUGLOVA ET AL. [9] and DEREVTSOV AND KASHINA [22]. ANDERSSON [4] uses higher moments of the velocity spectrum to reconstruct not only the solenoidal part but also irrotational parts of $\mathbf{f}$. In [109, 110] the author applies the concept of approximate inverse to the Doppler tomography problem yielding a method of filtered backprojection type.

As mentioned before, the applications of vector tomography problems are not confined to medicine. Sielschott [118] investigates the gas dynamics of a furnace using time-of-flight-measurements. Stefani, Gerbeth [121, 122] present a mathematical model to reconstruct velocities of electroconductive melts by measuring magnetic fields and electrical potentials which are induced by a disposed electric field. They deal with an integral equation which arises from Maxwell's equations and the Biot-Savart law and is not related to the Doppler transform.

The intention of this part of the book is to apply the concepts established in part 1 to the three-dimensional Doppler transform. In the first chapter we briefly outline how to get the Doppler transform of a velocity field by sending ultrasound waves. After that we introduce the semi-discrete Doppler transform corresponding to the fact that we have only a finite number of data available. Hence, the semi-discrete setup is relevant from the practical point of view. In the next chapter we provide all ingredients being necessary to apply the method of semi-discrete approximate inverse to the Doppler tomography problem. These include the mollifier operator $\mathbf{E}_d$ which is important for the convergence of the method and the interpolation operator $\Pi_{p,q,r}$ as well as the computation of reconstruction kernels. Chapter 8 is entirely dedicated to the convergence and stability analysis of the method. Since the null space of $\mathbf{D}$ contains the potential fields ∇p with vanishing boundary values we introduce defect correction methods in Chapter 9. One feature of these methods is that we only need the measured data to compute the correction term and not the approximate solution.