Part I

Inverse and Semi-discrete Problems

Many applications in natural science, industry, medicine and engineering can be concisely described by an operator equation of first kind

$$\mathbf{A}f = g\,. \tag{0.2}$$

Here, g can be seen as a set of measurement data and f is the quantity we are searching for. The mapping $\mathbf{A}$ tells how f and the data g are connected to each other. The properties of $\mathbf{A}$ have influence on the mathematical solution of (0.2). A problem like (0.2), that is observe g and find the solution f, is called an *inverse problem*.

The method of approximate inverse is a powerful and versatile tool for solving inverse problems. Articles as e.g. LOUIS [67], LOUIS, ABDULLAH [1], JONAS, LOUIS [49], LOUIS, SCHUSTER [75], RIEDER, SCHUSTER [102, 103], SCHUSTER [110] and SCHUSTER, QUINTO [115] prove this. The idea is to compute a smoothing f_γ of f. If f is a function this can be done by convolving f with e_γ which is a smooth function such as the Gaussian kernel. Then, using the duality of $\mathbf{A}$ and $\mathbf{A}^*$, f_γ can be calculated by evaluating inner products of the measured data g with a so-called reconstruction kernel υ_γ which is performed in an efficient and stable way. The approximate inverse represents a class of *regularization methods*, which by its flexibility allows an adjustment to the underlying problem at a high level. Further we will see how invariance properties of $\mathbf{A}$ enhance the efficacy of the resulting inversion scheme. In practical situations equation (0.2) often is not an appropriate setting. Instead of the 'complete' function g we maybe have only a finite number of observations, e.g. moments or point evaluations, of g at hand. Thus, a semi-discrete setting

$$\mathbf{A}_n f = g_n$$

is more suited to describe real-world problems. Here, $\mathbf{A}_n$ emerges from $\mathbf{A}$ and g_n from g by an application of the so-called *observation operator* which models the measurement procedure and contains all information about the measurement device, e.g. its geometry. In order to prove strong convergence we prefer to investigate the semi-discrete rather than a fully discrete setting, though the latter one seems to be more useful from a practical point of view. Given finitely many computed moments $\langle f, e_i\rangle$ of f we obtain an approximation to f using an appropriate interpolation mapping. All these issues are subject of the first part of the book.

Part I includes five chapters. The first chapter provides essential facts from the theory of regularization methods for inverse problems. The basic idea of the approximate inverse is then demonstrated for linear, bounded mappings between L^2-spaces in the second chapter along with the two-dimensional Radon transform as a first application. In Chapter 3 we extend the method to semi-discrete problems in arbitrary Hilbert spaces. We show that the approximate inverse in fact is a regularization method and present a rigorous convergence and stability analysis. Chapter 4 finally consists of the presentation of a framework for solving semi-discrete equations in distribution spaces,

which has important applications in SAR and SONAR. We furthermore sketch the idea of an error analysis for this case, too. We finish this part with some remarks collected in Chapter 5.