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The Method of Approximate Inverse: Theory and Applications

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Dedicated to Petra, for her patience, understanding, and love

Preface

Many questions and applications in natural science, engineering, industry or medical imaging lead to inverse problems, that is: given some measured data one tries to recover a searched for quantity. These problems are of growing interest in all these disciplines and thus there is a great need for modern and stable solvers for these problems. A prominent example of an inverse problem is the problem of computerized tomography: From measured X-ray attenuation coefficients one has to calculate densities in human tissue. Mathematically inverse problems often are described as operator equations of first kind

$$\mathbf{A}f = g\,, \tag{0.1}$$

where $\mathbf{A} : X \to Y$ is a bounded operator acting on appropriate topological spaces X and Y. In case of 2D computerized tomography the mapping $\mathbf{A}$ is given by the Radon transform. Typically these operators have unbounded inverses $\mathbf{A}^{-1}$, if they are invertible at all. For instance if $\mathbf{A}$ is compact with infinite dimensional range, then $\mathbf{A}^{-1}$ is not continuous. In case of Hilbert spaces X and Y the generalized inverse $\mathbf{A}^\dagger$ exists and has a dense domain. But $\mathbf{A}^\dagger$ is bounded if and only if the range of $\mathbf{A}$ is closed which is not satisfied for compact $\mathbf{A}$. In applications the exact data g is noise contaminated e.g. by the measurement process or discretization errors. Noisy data g^ε lead to an useless solution $f^\varepsilon = \mathbf{A}^{-1}g^\varepsilon$ or $f^\varepsilon = \mathbf{A}^\dagger g^\varepsilon$ in the sense that the error $f - f^\varepsilon$ is unacceptably large. Hence, the stable solution of equations like (0.1) with noisy right-hand side g^ε require regularization methods R_γ. The mappings R_γ are bounded operators which converge pointwise to the unbounded generalized inverse $\mathbf{A}^\dagger$. Many regularization techniques have been developed over the last decades such as the truncated singular value decomposition, the Tikhonov-Phillips regularization or iterative methods such as the Landweber method and the method of conjugate gradients (CG-method) to name only the most popular ones.

A powerful tool which subsumes a whole family of regularization techniques is the method of approximate inverse. This method uses the duality of

the operator and the spaces where it acts on. It calculates approximations to the exact solution by smoothing it with mollifiers which are approximations to Dirac's delta distribution and attenuate high frequencies contained in the solution. The method consists then of the evaluation of the measured data with so called reconstruction kernels. The reconstruction kernels themselves are solutions of an equation involving the dual operator and the mollifier and can be precomputed before the measurement process starts. A further feature of the method is its flexibility: it can be adjusted to the operator and the underlying spaces to improve the efficiency. The first idea of solving linear operator equations by mollifier methods arose in 1990 by Louis and Maass [71] and it was Louis, who published its first fundamental properties [66] and showed its regularization property [68]. Rieder and Schuster [101, 102] derived a setting of the method for operators between arbitrary Hilbert spaces and proved convergence with rates and stability. An extension of the method to spaces of distributions was done by Schuster, Quinto [115]. The article Schöpfer et al. [107] must be seen as a first step to realize this technique in Banach spaces.

This monograph contains a comprehensive outline of the theoretical aspects of the method of approximate inverse (Part I) as well as applications of the method to different inverse problems arising in medical imaging and non-destructive testing (Parts II-IV). Part I gives a brief introduction to inverse problems and regularization methods and introduces then the approximate inverse on spaces of square integrable functions, where the Radon transform serves as a first example. We then go one step further and present the abstract setup of solving semi-discrete operator equations between arbitrary Hilbert spaces by the method of approximate inverse. Semi-discrete operator equations are of wide interest since in practical applications only a finite number of measured data is available. Part I ends with an extension of the theory to spaces of distributions. Part II puts life into the theoretical considerations of Part I and demonstrates their transfer to the problem of 3D Doppler tomography. Doppler tomography belongs to the area of medical imaging and means the problem of recovering the velocity field of a moving fluid from ultrasonic Doppler measurements. It is outlined how the method of approximate inverse leads to a solver of filtered backprojection type on the one hand and can be involved in the construction of defect correction methods on the other hand. In SONAR (SOund in NAvigation and Radiation) and SAR (Synthetic Aperture Radar) the problem arises of inverting a spherical mean operator. If the center set consists of a hyperplane this operator can no longer be described as a bounded mapping between Hilbert or Banach spaces, but it extends to a linear, continuous mapping between spaces of tempered distributions. Part III of the book presents the extension of the method to distribution spaces and shows its performance when being applied to the spherical mean operator. Further applications such as X-ray diffractometry, which is a sort of non-destructive testing, thermoacoustic tomography, where the spherical mean operator is involved, too, but with spheres as center sets, and 3D

computerized tomography are the contents of Part IV. The book contains plenty of numerical results which prove that the method is well suited to cope with inverse problems in practical situations and each part is completed by a conclusion and future perspectives.

This monograph is an extended version of my habilitation thesis which I submitted at the Saarland University Saarbrücken (Germany) in 2004. The mathematical results contained therein would have been impossible to accomplish without some important people accompanying my scientific way now for many years. Thus, the first person I would like to thank is my teacher Prof. Dr. A.K. Louis who introduced me to the area of approximate inverse many years ago and who supported me all the time. Part II of the book was the result of an intensive collaboration with Prof. Dr. A. Rieder between 2000 and 2004 and I am still thankful for conveying his rich experience in approximation theory to me. Part III of the book was the result of an one year stay at Tufts University in Medford (USA) at the chair of Prof. Dr. E.T. Quinto, an acclaimed expert in integral geometry and numerical mathematics. I owe him many useful pointers with respect to the extension of the approximate inverse to distribution spaces and I will never forget his hospitality. I am further indebted to Dr. R. Müller for a very careful review of the manuscript.

Thomas Schuster Hamburg, January 2007

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