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The Method of Approximate Inverse: Theory and Applications

 Springer

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Dedicated to Petra, for her patience, understanding, and love

Preface

Many questions and applications in natural science, engineering, industry or medical imaging lead to inverse problems, that is: given some measured data one tries to recover a searched for quantity. These problems are of growing interest in all these disciplines and thus there is a great need for modern and stable solvers for these problems. A prominent example of an inverse problem is the problem of computerized tomography: From measured X-ray attenuation coefficients one has to calculate densities in human tissue. Mathematically inverse problems often are described as operator equations of first kind

$$\mathbf{A}f = g, \tag{0.1}$$

where $\mathbf{A} : X \rightarrow Y$ is a bounded operator acting on appropriate topological spaces X and Y . In case of 2D computerized tomography the mapping \mathbf{A} is given by the Radon transform. Typically these operators have unbounded inverses \mathbf{A}^{-1} , if they are invertible at all. For instance if \mathbf{A} is compact with infinite dimensional range, then \mathbf{A}^{-1} is not continuous. In case of Hilbert spaces X and Y the generalized inverse \mathbf{A}^\dagger exists and has a dense domain. But \mathbf{A}^\dagger is bounded if and only if the range of \mathbf{A} is closed which is not satisfied for compact \mathbf{A} . In applications the exact data g is noise contaminated e.g. by the measurement process or discretization errors. Noisy data g^ε lead to an useless solution $f^\varepsilon = \mathbf{A}^{-1}g^\varepsilon$ or $f^\varepsilon = \mathbf{A}^\dagger g^\varepsilon$ in the sense that the error $f - f^\varepsilon$ is unacceptably large. Hence, the stable solution of equations like (0.1) with noisy right-hand side g^ε require regularization methods R_γ . The mappings R_γ are bounded operators which converge pointwise to the unbounded generalized inverse \mathbf{A}^\dagger . Many regularization techniques have been developed over the last decades such as the truncated singular value decomposition, the Tikhonov-Phillips regularization or iterative methods such as the Landweber method and the method of conjugate gradients (CG-method) to name only the most popular ones.

A powerful tool which subsumes a whole family of regularization techniques is the method of approximate inverse. This method uses the duality of

the operator and the spaces where it acts on. It calculates approximations to the exact solution by smoothing it with mollifiers which are approximations to Dirac's delta distribution and attenuate high frequencies contained in the solution. The method consists then of the evaluation of the measured data with so called reconstruction kernels. The reconstruction kernels themselves are solutions of an equation involving the dual operator and the mollifier and can be precomputed before the measurement process starts. A further feature of the method is its flexibility: it can be adjusted to the operator and the underlying spaces to improve the efficiency. The first idea of solving linear operator equations by mollifier methods arose in 1990 by LOUIS AND MAASS [71] and it was Louis, who published its first fundamental properties [66] and showed its regularization property [68]. RIEDER AND SCHUSTER [101, 102] derived a setting of the method for operators between arbitrary Hilbert spaces and proved convergence with rates and stability. An extension of the method to spaces of distributions was done by SCHUSTER, QUINTO [115]. The article SCHÖPFER ET AL. [107] must be seen as a first step to realize this technique in Banach spaces.

This monograph contains a comprehensive outline of the theoretical aspects of the method of approximate inverse (Part I) as well as applications of the method to different inverse problems arising in medical imaging and non-destructive testing (Parts II-IV). Part I gives a brief introduction to inverse problems and regularization methods and introduces then the approximate inverse on spaces of square integrable functions, where the Radon transform serves as a first example. We then go one step further and present the abstract setup of solving semi-discrete operator equations between arbitrary Hilbert spaces by the method of approximate inverse. Semi-discrete operator equations are of wide interest since in practical applications only a finite number of measured data is available. Part I ends with an extension of the theory to spaces of distributions. Part II puts life into the theoretical considerations of Part I and demonstrates their transfer to the problem of 3D Doppler tomography. Doppler tomography belongs to the area of medical imaging and means the problem of recovering the velocity field of a moving fluid from ultrasonic Doppler measurements. It is outlined how the method of approximate inverse leads to a solver of filtered backprojection type on the one hand and can be involved in the construction of defect correction methods on the other hand. In SONAR (SOund in NAvigation and Radiation) and SAR (Synthetic Aperture Radar) the problem arises of inverting a spherical mean operator. If the center set consists of a hyperplane this operator can no longer be described as a bounded mapping between Hilbert or Banach spaces, but it extends to a linear, continuous mapping between spaces of tempered distributions. Part III of the book presents the extension of the method to distribution spaces and shows its performance when being applied to the spherical mean operator. Further applications such as X-ray diffractometry, which is a sort of non-destructive testing, thermoacoustic tomography, where the spherical mean operator is involved, too, but with spheres as center sets, and 3D

computerized tomography are the contents of Part IV. The book contains plenty of numerical results which prove that the method is well suited to cope with inverse problems in practical situations and each part is completed by a conclusion and future perspectives.

This monograph is an extended version of my habilitation thesis which I submitted at the Saarland University Saarbrücken (Germany) in 2004. The mathematical results contained therein would have been impossible to accomplish without some important people accompanying my scientific way now for many years. Thus, the first person I would like to thank is my teacher Prof. Dr. A.K. Louis who introduced me to the area of approximate inverse many years ago and who supported me all the time. Part II of the book was the result of an intensive collaboration with Prof. Dr. A. Rieder between 2000 and 2004 and I am still thankful for conveying his rich experience in approximation theory to me. Part III of the book was the result of an one year stay at Tufts University in Medford (USA) at the chair of Prof. Dr. E.T. Quinto, an acclaimed expert in integral geometry and numerical mathematics. I owe him many useful pointers with respect to the extension of the approximate inverse to distribution spaces and I will never forget his hospitality. I am further indebted to Dr. R. Müller for a very careful review of the manuscript.

Thomas Schuster

Hamburg, January 2007

Contents

Part I Inverse and Semi-discrete Problems

1	Ill-posed problems and regularization methods	5
2	Approximate inverse in L^2-spaces	11
2.1	The idea of approximate inverse	11
2.2	A first example: The Radon transform	17
3	Approximate inverse in Hilbert spaces	25
3.1	Semi-discrete operator equations	25
3.2	Convergence and stability	32
4	Approximate inverse in distribution spaces	39
4.1	Mollifier and reconstruction kernels in dual spaces of smooth functions	40
4.2	Dealing with semi-discrete equations	44
5	Conclusion and perspectives	49

Part II Application to 3D Doppler Tomography

6	A semi-discrete setup for Doppler tomography	55
7	Solving the semi-discrete problem	63
7.1	Definition of the operators $\Pi_{p,q,r}$ and \mathbf{E}_d	63
7.2	Computation of reconstruction kernels for \mathbf{D}_j	70
7.3	The method of approximate inverse for $\Psi_{p,q,r} \mathbf{D}$	76
8	Convergence and stability	81

XII Contents

9	Approaches for defect correction	89
9.1	Potentials as solutions of elliptic boundary value problems	90
9.2	The Neumann problem	93
9.2.1	A boundary element method for the Neumann problem	93
9.2.2	The computation of the Newton potentials	94
9.2.3	Numerical results	100
9.3	The Dirichlet problem	101
10	Conclusion and perspectives	105

Part III Application to the spherical mean operator

11	The spherical mean operator	111
11.1	Spherical means in SONAR and SAR	111
11.2	Properties of the spherical mean operator	113
11.3	Approximate inverse for \mathbf{M}	118
12	Design of a mollifier	123
13	Computation of reconstruction kernels	133
14	Numerical experiments	139
15	Conclusion and perspectives	145

Part IV Further Applications

16	Approximate inverse and X-ray diffractometry	151
16.1	X-ray diffractometry	151
16.2	Approximate inverse for the Laplace transform	153
16.3	A solution scheme for the X-ray diffractometry problem	161
17	A filtered backprojection algorithm for thermoacoustic computerized tomography (TCT)	165
17.1	Thermoacoustic computerized tomography (TCT)	165
17.2	An inversion method for the spherical geometry	168
17.3	Numerical results	175
18	Computation of reconstruction kernels in 3D computerized tomography	181

19 Conclusion and perspectives	187
References	189
Index	197