Lecture Notes in Mathematics 1898

Horst Reinhard Beyer

Beyond Partial Differential Equations

On Linear and Quasi-Linear Abstract Hyperbolic Evolution Equations

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Dedicated to God the Father

Preface

Semigroup Theory uses abstract methods of Operator Theory to treat initial boundary value problems for linear and nonlinear equations that describe the evolution of a system. Due to the generality of its methods, the class of systems that can be treated in this way exceeds by far those described by equations containing only local operators induced by partial derivatives, i.e., PDEs. In particular, that class includes the systems of Quantum Theory.

Another important application of semigroup methods is in field quantization. Simple examples are given by the cases of free fields in Minkowski spacetime like Klein-Gordon fields, the Dirac field and the Maxwell field, whose field equations are given by systems of linear PDEs. The second quantization of such a field replaces the field equation by a Schrödinger equation whose Hamilton operator is given by the second quantization of a non-local function of a self-adjoint linear operator. That operator generates the semigroup given by the time-development of the solutions of the field equation corresponding to arbitrary initial data as a function of time. More generally, in these cases the structures used in the formulation of a well-posed abstract initial value problem for the field equation also provide the mathematical framework for the quantization of the field. Quantum Theory is an abstract theory, therefore it should be expected that only an abstract approach to classical field equations using methods from Operator Theory is capable of providing the appropriate structures for quantization in the less simple cases of nonlinear fields, like the gravitational field described by Einstein's field equations.

A demonstration of the strength of semigroup methods can be seen in the first rigorous proof of well-posedness (local in time) of the initial value problem for quasi-linear symmetric hyperbolic systems by T. Kato in 1975 in [110]. This result is a particular application of a theorem on the well-posedness of the initial value problem for abstract quasi-linear equations[1], which has been successfully applied also to Einstein's equation's [102], the Navier-Stokes equations, the equations of Magnetohydrodynamics [109] and more. To my knowledge, there is no other approach to quasi-linear equations leading to a theorem of such generality.

[1] See Theorem 11.0.7 below.

The semigroup approach goes beyond that of a tool for deciding the well-posedness of initial boundary value problems. For a autonomous nonlinear equation the important question of the linearized stability of a particular solution leads on a spectral problem for the operator generating the semigroup given by the time-development of the solutions of the linearized equation around that solution corresponding to arbitrary initial data as a function of time.[2] These methods also provide, for autonomous linear equations, a representation of the solution of the initial value problem as an integral over the resolvent of the infinitesimal generator of the associated semigroup.[3] The resolvent operators can often be represented in the form of integral operators with kernels which are defined in terms of special functions. This is not only true in simple cases where the generator is a partial differential operator with constant coefficients, but also in a number of cases involving non-constant coefficients. In this way, an integral representation of the solution of the initial value problem is achieved for all such cases.[4] Finally, semigroup methods provide a framework which is general enough to include numerical forms of evolution equations, opening the possibility of computing true error estimates to the exact solution, rather than residual errors.

This should not give the impression that the semigroup approach could replace all 'hard' analysis facts. Instead, it reduces such application to a bare minimum, which gives the approach its efficiency.[5] For instance, results from harmonic analysis or the theory of singular integral operators have applications in so called 'commutator estimates' where the commutator of an intertwining operator with the principal part of a partial differential operator, usually two unbounded operators, has to be estimated. To achieve most general results, it is often necessary to choose intertwining operators as non-local operators.

These methods are especially attractive *if not inevitable* for theoretical physicists in view of their comprehension of classical physics and quantum physics. In addition, their efficiency is of advantage in view of time restrictions in the mathematics education of physicists. In spite of their power, efficiency and versatility, semigroup methods are surprisingly little used in theoretical physics.[6,7] This appears to be related to two misconceptions.

First, because of the requirements of Special Relativity and General Relativity, current problems in fundamental theoretical physics necessarily lead on hyperbolic

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2 For instance, see Chapter 5.4.

3 Roughly speaking, for a precise statement see, e.g., Chapter 4.3.

4 See, e.g., [25].

5 But it is the belief of the author that the necessity of the use of 'hard' to achieve analytical facts in the solution of a problem indicates that its structure has not yet been fully understood.

6 Paradoxically, in some sense, one could also hold the opposite view that these methods have been used for a long time in theoretical physics, mostly without realizing that they are rooted in in Spectral Theory.

7 In addition, the author is not aware of a single introduction to PDE based solely on Semigroups/Operator Theory, although this would have been possible even before the appearance of classical introductions like [20] that use less general methods.

partial differential equation systems ('hyperbolic problems'). Standard texts on semigroups of linear operators mainly focus on applications to parabolic partial differential equation systems ('parabolic problems'). Differently to the hyperbolic case, this leads to the consideration of strongly continuous semigroups which are in addition analytic. Apparently, the consideration of parabolic problems originates from a focus on engineering applications. Engineering sciences predominantly apply classical Newtonian physics where signal propagation speeds are not limited by the speed of light in vacuum as this is the case in Special Relativity/General Relativity. For example, the evolution of a compactly supported temperature field in space at time $t = 0$ by the parabolic heat equation leads to a temperature field that has no compact support for every $t > 0$. As consequence, in the evolution signal propagation speeds occur that exceed the speed of light in vacuum, and hence the equation is incompatible with Special Relativity. The same is also true for Schrödinger equations.[8] The evolution of hyperbolic partial differential equations systems preserves the compactness of the support of the data under time evolution. To my experience, the focus on engineering applications has lead to the quite common misconception among physicists, and to some extent also among mathematicians working in the field of partial differential equations, that semigroup methods cannot be applied to hyperbolic problems.

Second, to my experience, another common misconception is that semigroups of linear operators can only be applied to systems of *linear* partial differential equations. This might be influenced by the fact that most standard texts on semigroups of linear operators, indeed, focus mainly on such applications.

As a consequence, the course should lead as rapidly as possible from autonomous linear equations to the nonlinear (quasi-linear) equations which are now seen in the hyperbolic problems emanating from current physics. Particular stress is on wave equations and Hermitian hyperbolic systems. The last cover the equations describing interacting fields in physics and therefore the major part of nonlinear equations occurring in fundamental physics. Throughout the course applications to problems from current relativistic ('hyperbolic') physics are provided, which display the potential of the methods in the solution of current problems in physics. These include problems from black hole physics, the formulation of outgoing boundary conditions for wave equations and the treatment of additional constraints. The last two are important current problems in the numerical evolution of Einstein's field equations for the gravitational field. Some of the examples contain new unpublished results of the author. To my knowledge, the major part of the material in the second part of the notes, including non-autonomous and quasi-linear Hermitian hyperbolic systems, has appeared only inside research papers.

The orientation of this course towards abstract quasi-linear evolution equations made it necessary to omit a number of topics that were not directly important for achieving its goals. On the other hand, texts on semigroups of linear operators that consider those topics are available [39,47,52,57,90,99,106,120,168,179,224]. Also,

[8] This was the reason for the development of Quantum Field Theory.

there is a theory of nonlinear semigroups that largely parallels that of semigroups of linear operators [15, 19, 88, 138, 146, 167].

This course assumes some basic knowledge of Functional Analysis which can be found, for instance, in the first volume of [179]. Some examples assume more specialized knowledge of properties of self-adjoint linear operators in Hilbert spaces which can be found, for instance, in the second volume of [179]. In addition, some applications assume basic knowledge of Sobolev spaces of the L^2-type as is provided, for instance, in [217]. Otherwise this course is self-contained. The material is presented in as compressed a form as possible and its results are formulated in view towards applications. In general, theorems contain their full set of assumptions, so that a study of their environment is not necessary for their understanding. For this reason and to limit the size of this text, shorter definitions appear as part of theorems. In addition, the abstract theory and its applications appear in separate chapters to allow the reader to estimate the necessary time to acquire the theory.

Chapters 2–5 constitute the notes of a course given at the Department of Mathematics of the Louisiana State University in Baton Rouge in Spring 2005. They provide a basic introduction into the properties of strongly continuous semigroups on Banach spaces and applications to autonomous linear hyperbolic systems of PDEs from General Relativity and Astrophysics. The theoretical part is kept to a minimum with a view to applications in the field of hyperbolic PDEs. It is formulated in a way which is expected to be natural to readers with a knowledge of the spectral theory of self-adjoint linear operators in Hilbert spaces. An exception, to this restriction to the minimum in these chapters, is the treatment of the integration of Banach space-valued maps which is more detailed than is usual in most other comparable texts. This is due to the fact that such integration is a basic tool which is usually not covered in standard Functional Analysis courses, at least not to an extent needed in the study of semigroups of operators. Otherwise, the theoretical material is standard and for this reason only few references to literature are given. For more comprehensive introductions into the theory of semigroups of linear operators that give a more exhaustive list of references, the reader is referred, for instance, to [57, 90, 99, 168].

Chapters 6–12 are the notes of a subsequent course at the same place in Spring 2006. They introduce into the field of abstract evolution equations with applications to non-autonomous linear and quasi-linear hyperbolic systems of PDEs. This second part of the notes follows closely the late Tosio Kato's 1993 paper *'Abstract evolution equations, linear and quasilinear, revisited'* in Proceedings of the International Conference in Memory of Professor Kosaku Yosida held at RIMS, Kyoto University, Japan, July 29-Aug. 2, 1991, [114]. Those results are more general than Kato's well-known older results[9] [107–109] in that they don't assume special properties of the underlying Banach spaces. Proofs of Kato's results are added along with detailed examples displaying their application to problems in current relativistic physics. In a few places, Lemmata were added to his outline that appeared necessary for those proofs.

[9] Those results assume the reflexivity and in some places also the separability of the underlying Banach spaces.

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