Lecture Notes in Mathematics 1959

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Weight Filtrations on Log Crystalline Cohomologies of Families of Open Smooth Varieties

Springer

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ISBN: 978-3-540-70564-2 e-ISBN: 978-3-540-70565-9
DOI: 10.1007/978-3-540-70565-9

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2008932186

Mathematics Subject Classification (2000): 14F30 (Primary), 14F40, 13K05 (Secondary)

Cover design: SPi Publishing Services

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface

The main goal of this book is to construct a theory of weights for the log crystalline cohomologies of families of open smooth varieties in characteristic $p > 0$. This is a p-adic analogue of the theory of the mixed Hodge structure on the cohomologies of open smooth varieties over $\mathbb{C}$ developed by Deligne in [23]. We also prove the fundamental properties of the weight-filtered log crystalline cohomologies such as the p-adic purity, the functoriality, the weight-filtered base change theorem, the weight-filtered Künneth formula, the convergence of the weight filtration, the weight-filtered Poincaré duality and the E_2-degeneration of p-adic weight spectral sequences. One can regard some of these results as the logarithmic and weight-filtered version of the corresponding results of Berthelot in [3] and K. Kato in [54].

Following the suggestion of one of the referees, we have decided to state some theorems on the weight filtration and the slope filtration on the rigid cohomology of separated schemes of finite type over a perfect field of characteristic $p > 0$. This is a p-adic analogue of the mixed Hodge structure on the cohomologies of separated schemes of finite type over $\mathbb{C}$ developped by Deligne in [24]. The detailed proof for them is given in another book [70] by the first-named author.

We have to assume that the reader is familiar with the basic premises and properties of log schemes ([54], [55]) and (log) crystalline cohomologies ([3], [11], [54]). We hope that the findings in this book will serve as a role as a first step to understanding the rich structures which p-adic cohomology theory should have.

Tokyo — *Yukiyoshi Nakkajima*
January 2008 — *Atsushi Shiho*

Acknowledgements

We would like to thank K. Bannai, as the genesis of this work was a discussion with him in Azumino in 2000. Three lectures of P. Berthelot in Dwork trimester in 2001 [8] were also very helpful for us. Most key ideas in §1.1 and §1.2 can be attributed to him. We are deeply grateful to him for his permission to include some of his findings in this book. We would also like to thank to K. Sato for a useful discussion. We gave some proofs in this book during both authors' stay at Padova University and during the first-named author's stay at Université de Rennes 1. Finally, we are very grateful to B. Chiarellotto, B. Le Stum and the universities for their hospitality.

Contents

Introduction

Though they are vague, we have the following dreams as in [37], [22] and [25]:

(1) Cohomologies in characteristic 0 of algebro-geometric objects in any characteristic have remarkable increasing filtrations, which are called weight filtrations.

(2) They are motivic.

(3) They are constructible sheaves. Sometimes they are, in fact, smooth sheaves.

(4) They are compatible with canonical operations, e.g., base change, Künneth formula, Poincaré duality.

(5) They are functorial: certain classes of morphisms (e.g., the induced morphisms by the morphisms of algebro-geometric objects) are strictly compatible with them.

In this book, for a family of open smooth varieties in characteristic $p > 0$, we discuss the p-adic aspects of (1), (3), (4), (5) and the following new aspect:

(6) In some cases, they grow and they are rigid as Grothendieck said for crystalline sheaves.

Here we assume that the family is the complement of a relative simple normal crossing divisor on a family of smooth varieties.

Before explaining our results, we recall Deligne's result on the weight filtration on the higher direct image of $\mathbb{Q}$ by a morphism from a family of open smooth algebraic varieties with good compactifications to a base scheme over the complex number field $\mathbb{C}$ ([23], [25]).

Let U be a smooth variety over $\mathbb{C}$. Let X be a smooth variety over $\mathbb{C}$ with a simple normal crossing divisor D such that $U = X \setminus D$. Let $j\colon U \xrightarrow{\subset} X$ be the natural open immersion. Set $D^{(0)} := X$ and, for a positive integer k, let $D^{(k)}$ be the disjoint union of all k-fold intersections of the different irreducible components of D. Let $P := \{P_k\}_{k\in\mathbb{Z}}$ be the weight filtration on the sheaf $\Omega^i_{X/\mathbb{C}}(\log D)$ ($i \in \mathbb{N}$) of the logarithmic differential forms on X which

Y. Nakkajima, A. Shiho, *Weight Filtrations on Log Crystalline Cohomologies of Families of Open Smooth Varieties.* Lecture Notes in Mathematics 1959,

is obtained by counting the number of the logarithmic poles of local sections of $\Omega^i_{X/\mathbb{C}}(\log D)$. Let $a^{(k)}\colon D^{(k)} \longrightarrow X$ $(k \in \mathbb{N})$ be the natural morphism of schemes over $\mathbb{C}$. Then we have the following isomorphism of complexes (the Poincaré residue isomorphism):

$$\text{(0.0.0.1)} \qquad \mathrm{Res}\colon \mathrm{gr}^P_k \Omega^\bullet_{X/\mathbb{C}}(\log D) \xrightarrow{\sim} a^{(k)}_*(\Omega^{\bullet -k}_{D^{(k)}/\mathbb{C}} \otimes_{\mathbb{Z}} \varpi^{(k)}(D/\mathbb{C})(-k)).$$

Here $\varpi^{(k)}(D/\mathbb{C})$ is the orientation sheaf of $D^{(k)}/\mathbb{C}$; $\varpi^{(k)}(D/\mathbb{C})(-k) := \epsilon^k_{\mathbb{Z}}$ in [23]. By using the isomorphism (0.0.0.1), we have the following spectral sequence

(0.0.0.2)
$$E_1^{-k,h+k} = H^{h-k}(D^{(k)}, \Omega^\bullet_{D^{(k)}/\mathbb{C}} \otimes_{\mathbb{Z}} \varpi^{(k)}(D/\mathbb{C}))(-k) \Longrightarrow H^h(X, \Omega^\bullet_{X/\mathbb{C}}(\log D)).$$

Moreover we have the following isomorphisms in the filtered derived category of bounded below filtered complexes of $\mathbb{C}_{X_{\mathrm{an}}}$-modules:

$$\text{(0.0.0.3)} \qquad (\Omega^\bullet_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), P) \xleftarrow{\sim} (\Omega^\bullet_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), \tau) \xrightarrow{\sim}$$
$$(j_{\mathrm{an}*}(\Omega^\bullet_{U_{\mathrm{an}}/\mathbb{C}}), \tau) \xrightarrow{\sim} (Rj_{\mathrm{an}*}(\mathbb{C}_{U_{\mathrm{an}}}), \tau) = (Rj_{\mathrm{an}*}(\mathbb{Z}_{U_{\mathrm{an}}}) \otimes_{\mathbb{Z}} \mathbb{C}, \tau),$$

where $\tau := \{\tau_k\}_{k \in \mathbb{Z}}$ is the canonical filtration. By using the exponential sequence on U_{an} and the cup product, we have the purity isomorphism

$$\text{(0.0.0.4)} \qquad R^k j_{\mathrm{an}*}(\mathbb{Z}_{U_{\mathrm{an}}}) \xleftarrow{\sim} a^{(k)}_{\mathrm{an}*}(\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k) \quad (k \in \mathbb{N})$$

(cf. [58, (1.5.1)]). By the same calculation as that in [69, (3.3)], the following morphism

$$\text{(0.0.0.5)} \qquad \begin{aligned} & a^{(k)}_{\mathrm{an}*}(\mathbb{C}_{(D_{\mathrm{an}})^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k) \\ \xleftarrow[\sim]{(0.0.0.4)\otimes\mathbb{C}} \quad & R^k j_{\mathrm{an}*}(\mathbb{C}_{U_{\mathrm{an}}}) \\ \overset{(0.0.0.3)}{=} \quad & \mathrm{gr}^P_k \Omega^{\bullet +k}_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}) \\ \xrightarrow[\sim]{\mathrm{Res}} \quad & a^{(k)}_{\mathrm{an}*}(\Omega^\bullet_{(D_{\mathrm{an}})^{(k)}/\mathbb{C}} \otimes_{\mathbb{Z}} \varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k) \\ = \quad & a^{(k)}_{\mathrm{an}*}(\mathbb{C}_{(D_{\mathrm{an}})^{(k)}} \otimes_{\mathbb{Z}} \varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k) \end{aligned}$$

is equal to the multiplication by $(-1)^k$. Hence we use the following isomorphism

(0.0.0.6)
$$R^k j_{\mathrm{an}*}(\mathbb{Z}_{U_{\mathrm{an}}}) \xrightarrow[\sim]{(0.0.0.4)} a^{(k)}_{\mathrm{an}*}(\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k) \xrightarrow[\sim]{(-1)^k} a^{(k)}_{\mathrm{an}*}(\varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k)$$

instead of (0.0.0.4). The isomorphism (0.0.0.6) is equal to the isomorphism in [23, (3.1.9)]. Using the Leray spectral sequence for two functors $j_{\mathrm{an}*}$ and $\Gamma(X_{\mathrm{an}}, ?)$, using the isomorphism (0.0.0.6) and renumbering $E_r^{-k,h+k} := E_{r+1}^{h-k,k}$, we have the following spectral sequence

$$(0.0.0.7) \quad E_1^{-k,h+k} = H^{h-k}((D_{\mathrm{an}})^{(k)}, \varpi^{(k)}(D_{\mathrm{an}}/\mathbb{C}))(-k) \Longrightarrow H^h(U_{\mathrm{an}}, \mathbb{Z}).$$

If X is proper, one obtains the weight filtration on $H^h(U_{\mathrm{an}}, \mathbb{Z})$ by the spectral sequence (0.0.0.7); if X is proper, (0.0.0.2) is equal to (0.0.0.7)$\otimes_{\mathbb{Z}}\mathbb{C}$ by GAGA and the Poincaré lemma.

In fact, the existence of the weight filtration above modulo torsion has been generalized to the case of a family in characteristic 0 by Deligne as follows. Let $f\colon X \longrightarrow S$ be a proper smooth morphism of schemes of finite type over $\mathbb{C}$. Let D be a relative simple normal crossing divisor on X over S. Set $U := X \setminus D$, and let f also denote the structural morphism $f\colon U \longrightarrow S$. Then $R^h f_{\mathrm{an}*}(\mathbb{Q}_{U_{\mathrm{an}}})$ is a local system ([21, II (6.14)]), and there exists a filtration P on $R^h f_{\mathrm{an}*}(\mathbb{Q}_{U_{\mathrm{an}}})$ by sub local systems such that the induced filtration P_s on the stalk $R^h f_{\mathrm{an}*}(\mathbb{Q}_{U_{\mathrm{an}}})_s = H^h((U_{\mathrm{an}})_s, \mathbb{Q})$ $(s \in S_{\mathrm{an}})$ ([loc. cit.]) is obtained from the spectral sequence (0.0.0.7) ([25]).

Now let us turn to the case of characteristic $p > 0$.

Let $(S, \mathcal{I}, \gamma)$ be a PD-scheme with a quasi-coherent ideal sheaf $\mathcal{I}$. Set $S_0 := \underline{\mathrm{Spec}}_S(\mathcal{O}_S/\mathcal{I})$. Assume that p is locally nilpotent on S. Let $f\colon X \longrightarrow S_0$ be a smooth scheme with a relative simple normal crossing divisor D on X over S_0 (by abuse of notation, we denote by the same symbol f the composite morphism $X \xrightarrow{f} S_0 \xrightarrow{\subset} S$). Then the pair (X, D) of S_0-schemes defines an fs(=fine and saturated) log scheme $(X, M(D))$ over S_0 (§2.1 below, cf. [54]) in the sense of Fontaine-Illusie-Kato and $(X, D)/S$ defines a log crystalline topos $((X, \widetilde{M(D)})/S)^{\log}_{\mathrm{crys}}$ ([54], cf. [29]). By abuse of notation, we often denote $(X, M(D))$ by (X, D). Once we obtain the topos $((\widetilde{X, D})/S)^{\log}_{\mathrm{crys}}$, we can use powerful techniques of [42] and many techniques of [3] (cf. [54]). Let $\mathcal{O}_{(X,D)/S}$ be the structure sheaf in $((\widetilde{X, D})/S)^{\log}_{\mathrm{crys}}$ and $\mathcal{O}_{D^{(k)}/S}$ the structure sheaf in the classical crystalline topos $(\widetilde{D^{(k)}/S})_{\mathrm{crys}}$. (See (2.2.13.2) and (2.2.15) below for the precise definition of $D^{(k)}$ for a nonnegative integer k.) Let $f_{(X,D)/S}\colon ((\widetilde{X, D})/S)^{\log}_{\mathrm{crys}} \longrightarrow \widetilde{S}_{\mathrm{zar}}$ and $f_{D^{(k)}/S}\colon (\widetilde{D^{(k)}/S})_{\mathrm{crys}} \longrightarrow \widetilde{S}_{\mathrm{zar}}$ be the natural morphisms of topoi. Then one of our main results in this book is to show the existence of the following functorial spectral sequence:

$$(0.0.0.8) \qquad \begin{aligned} E_1^{-k,h+k} &= R^{h-k} f_{D^{(k)}/S*}(\mathcal{O}_{D^{(k)}/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D/S))(-k) \\ &\Longrightarrow R^h f_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}). \end{aligned}$$

Here $\varpi^{(k)}_{\mathrm{crys}}(D/S)$ is the crystalline orientation sheaf of D/S in $(\widetilde{D^{(k)}/S})_{\mathrm{crys}}$ which will be defined in §2.2. If S_0 is of characteristic $p > 0$, then the relative

Frobenius morphism $F\colon (X,D) \longrightarrow (X',D')$ over S_0 induces the relative Frobenius morphism $F^{(k)}\colon D^{(k)} \longrightarrow D'^{(k)} = D^{(k)\prime}$. Let $a^{(k)}\colon D^{(k)} \longrightarrow X$ and $a^{(k)\prime}\colon D^{(k)\prime} \longrightarrow X'$ be the natural morphisms. We define the relative Frobenius action

$$\Phi^{(k)}\colon a^{(k)\prime}_{\mathrm{crys}*}\varpi^{(k)}_{\mathrm{crys}}(D'/S) \longrightarrow F_{\mathrm{crys}*}a^{(k)}_{\mathrm{crys}*}\varpi^{(k)}_{\mathrm{crys}}(D/S)$$

as the identity under the natural identification

$$\varpi^{(k)}_{\mathrm{crys}}(D'/S) \xrightarrow{\sim} F^{(k)}_{\mathrm{crys}*}\varpi^{(k)}_{\mathrm{crys}}(D/S).$$

Then (0.0.0.8) is compatible with the Frobenius action. We call (0.0.0.8) the *preweight spectral sequence* of $(X,D)/(S,\mathcal{I},\gamma)$. Here, as noted in [68], we use the terminology "preweight" instead of the terminology "weight" since $\mathcal{O}_S$ is a sheaf of torsion modules (and hence $R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$ is also). If S_0 is the spectrum of a perfect field κ of characteristic $p>0$ and if S is the spectrum of the Witt ring W_n of finite length $n>0$ of κ, then (0.0.0.8) is canonically isomorphic to the following preweight spectral sequence

$$E_1^{-k,h+k} = H^{h-k}((\widetilde{D^{(k)}/W_n})_{\mathrm{crys}}, \mathcal{O}_{D^{(k)}/W_n} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\mathrm{crys}}(D/W_n))(-k)$$
$$\Longrightarrow H^h_{\text{log-crys}}((X,D)/W_n),$$

essentially constructed in [65] and [68].

Let V be a complete discrete valuation ring of mixed characteristics with perfect residue field of characteristic $p>0$. Then we can also construct the spectral sequence (0.0.0.8) when S is a p-adic formal V-scheme in the sense of [74]. In this case, we call (0.0.0.8) the *p-adic weight spectral sequence* of $(X,D)/S$ and the induced filtration on $R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$ by (0.0.0.8) the *weight filtration* on $R^hf_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$.

Let us return to the case where $(S,\mathcal{I},\gamma)$ is a PD-scheme as above; especially, a prime number p is locally nilpotent on S. Let $\mathcal{O}_{X/S}$ be the structure sheaf in the classical crystalline topos $(\widetilde{X/S})_{\mathrm{crys}}$. Let $u_{(X,D)/S}\colon ((\widetilde{X,D})/S)^{\log}_{\mathrm{crys}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$ (resp. $u_{X/S}\colon (\widetilde{X/S})_{\mathrm{crys}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$) be the natural (resp. classical) projection. Then $u_{X/S}$ induces a morphism $u_{X/S}\colon ((\widetilde{X/S})_{\mathrm{crys}}, \mathcal{O}_{X/S}) \longrightarrow (\widetilde{X}_{\mathrm{zar}}, f^{-1}(\mathcal{O}_S))$ of ringed topoi. Let $u_{D^{(k)}/S}\colon (\widetilde{D^{(k)}/S})_{\mathrm{crys}} \longrightarrow \widetilde{D^{(k)}}_{\mathrm{zar}}$ be also the classical projection. Let $\epsilon_{(X,D)/S}\colon ((\widetilde{X,D})/S)^{\log}_{\mathrm{crys}} \longrightarrow (\widetilde{X/S})_{\mathrm{crys}}$ be the forgetting log morphism induced by the morphism $(X,M(D)) \longrightarrow (X,\mathcal{O}_X^*)$ of log schemes. Then $u_{X/S}\circ\epsilon_{(X,D)/S} = u_{(X,D)/S}$. Let $Q_{X/S}\colon (\widetilde{X/S})_{\mathrm{Rcrys}} \longrightarrow (\widetilde{X/S})_{\mathrm{crys}}$ be a morphism of topoi defined in [3, IV (2.1.1)]. Then we have a morphism $Q_{X/S}\colon ((\widetilde{X/S})_{\mathrm{Rcrys}}, Q^*_{X/S}(\mathcal{O}_{X/S})) \longrightarrow ((\widetilde{X/S})_{\mathrm{crys}}, \mathcal{O}_{X/S})$ of ringed topoi. Set $\overline{u}_{X/S} := u_{X/S}\circ Q_{X/S}\colon ((\widetilde{X/S})_{\mathrm{Rcrys}}, Q^*_{X/S}(\mathcal{O}_{X/S})) \longrightarrow (\widetilde{X}_{\mathrm{zar}}, f^{-1}(\mathcal{O}_S))$ as in [3].

To construct (0.0.0.8) and to prove the functoriality of it, we define two filtered complexes

(0.0.0.9)
$$(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P) := (E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),\{P_kE_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S})\}_{k\in\mathbb{Z}}) \in \mathrm{D}^+\mathrm{F}(\mathcal{O}_{X/S}),$$

(0.0.0.10)
$$(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P) := (E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),\{P_kE_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})\}_{k\in\mathbb{Z}}) \in \mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$$

and construct two other filtered complexes

(0.0.0.11)
$$(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P) := (C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),\{P_kC_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S})\}_{k\in\mathbb{Z}}) \in \mathrm{D}^+\mathrm{F}(Q^*_{X/S}(\mathcal{O}_{X/S})),$$

(0.0.0.12)
$$(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P) := (C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),\{P_kC_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})\}_{k\in\mathbb{Z}}) \in \mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S)).$$

Here $\mathrm{D}^+\mathrm{F}(\mathcal{O}_{X/S})$, $\mathrm{D}^+\mathrm{F}(Q^*_{X/S}(\mathcal{O}_{X/S}))$ and $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$ are the filtered derived categories of the bounded below filtered $\mathcal{O}_{X/S}$-modules, $Q^*_{X/S}(\mathcal{O}_{X/S})$-modules and $f^{-1}(\mathcal{O}_S)$-modules, respectively.

The definitions of $(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P)$ and $(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$ are as follows:

$$(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P) := (R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}),\tau),$$
$$(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P) := Ru_{X/S*}(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P).$$

(Here τ denotes the canonical filtration (§2.7).) Note that they are functorial with respect to (X,D). (This is not the case for $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)$.)

In a simple case we soon give the definition of $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)$ in (0.0.0.14) below. In the general case we give it in the text. The filtered complex $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$ is, by definition, $R\overline{u}_{X/S*}(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)$. We call $(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P)$ and call $(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$ the *preweight-filtered vanishing cycle crystalline complex* and the *preweight-filtered vanishing cycle zariskian complex* of $(X,D)/(S,\mathcal{I},\gamma)$, respectively. We also call $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)$ the *preweight-filtered restricted crystalline complex* of $(X,D)/(S,\mathcal{I},\gamma)$ and call $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$ the *preweight-filtered zariskian complex* of $(X,D)/(S,\mathcal{I},\gamma)$, respectively. The main theme of this book is to investigate fundamental properties of $(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P)$, $(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$, $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)$ and $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$. They enjoy the following properties:

(0.0.0.13): $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)\overset{\sim}{\longleftarrow} Q^*_{X/S}(E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S}),P)$; $\{P_kE_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S})\}_{k\in\mathbb{Z}}$ is an "increasing filtration" on $E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S})$ which is finite locally on X such that $P_{-1}E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S})=0$, $Q^*_{X/S}P_0E_{\mathrm{crys}}(\mathcal{O}_{(X,D)/S})\overset{\sim}{\longleftarrow} Q^*_{X/S}(\mathcal{O}_{X/S})$ and $C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S})\overset{\sim}{\longleftarrow} Q^*_{X/S}R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$.

(0.0.0.14): Let $\Delta:=\{D_\lambda\}_{\lambda\in\Lambda}$ be a decomposition of D by smooth components of D: $D=\bigcup_{\lambda\in\Lambda}D_\lambda$ and each D_λ is smooth over S_0. If (X,D) has an admissible immersion $(X,D)\overset{\subset}{\longrightarrow}(\mathcal{X},\mathcal{D})$ over S with respect to Δ (see (2.1.10) below for the definition of the admissible immersion), then

$$(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)\simeq(Q^*_{X/S}L_{X/S}(\Omega^\bullet_{\mathcal{X}/S}(\log\mathcal{D})),\\ \{Q^*_{X/S}L_{X/S}(P_k\Omega^\bullet_{\mathcal{X}/S}(\log\mathcal{D}))\}_{k\in\mathbb{Z}}),$$

where $L_{X/S}$ is the classical linearization functor for $\mathcal{O}_{\mathcal{X}}$-modules ([3, IV 3], [11, §6]).

(0.0.0.15): $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)\overset{\sim}{\longleftarrow}(E_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$.

(Hence $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)$ is functorial with respect to (X,D).) In particular,

(0.0.0.16): $\{P_kC_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})\}_{k\in\mathbb{Z}}$ is an "increasing filtration" on $C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})$ which is finite locally on X such that $P_{-1}C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})=0$, $P_0C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})\overset{\sim}{\longleftarrow} Ru_{X/S*}(\mathcal{O}_{X/S})$, and $C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})\overset{\sim}{\longleftarrow} Ru_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$, and

(0.0.0.17): If (X,D) has an admissible immersion $(X,D)\overset{\subset}{\longrightarrow}(\mathcal{X},\mathcal{D})$ over S with respect to $\Delta=\{D_\lambda\}_{\lambda\in\Lambda}$,

$$(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}),P)\simeq(\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}\Omega^\bullet_{\mathcal{X}/S}(\log\mathcal{D}),\{\mathcal{O}_{\mathfrak{D}}\otimes_{\mathcal{O}_{\mathcal{X}}}P_k\Omega^\bullet_{\mathcal{X}/S}(\log\mathcal{D})\}_{k\in\mathbb{Z}}),$$

where $\mathfrak{D}$ is the PD-envelope of the immersion $X\overset{\subset}{\longrightarrow}\mathcal{X}$ over $(S,\mathcal{I},\gamma)$.

(0.0.0.18): $\mathrm{gr}^P_k(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}))=Q^*_{X/S}a^{(k)}_{\mathrm{crys}*}(\mathcal{O}_{D^{(k)}/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{crys}}(D/S))\{-k\}$, where $\{-k\}$ is the shift which will be defined in the Convention (1) below (note that we do not consider the Tate twist $(-k)$ on the right hand side of (0.0.0.18) because the functor $Q^*_{X/S}$ appears on the right hand side (In [3, IV (2.5)] Berthelot has noted that the restricted crystalline topos does not have the functoriality in general).).

(0.0.0.19): $\mathrm{gr}^P_k(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}))=a^{(k)}_{\mathrm{zar}*}(Ru_{D^{(k)}/S*}(\mathcal{O}_{D^{(k)}/S})\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{zar}}(D/S_0))(-k)\{-k\}$, where $\varpi^{(k)}_{\mathrm{zar}}(D/S_0)$ is the zariskian orientation sheaf of D/S_0 which will be defined in §2.2. Here, see §2.9 for the meaning of the Tate twist $(-k)$.

(0.0.0.20): $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),\tau)\overset{\sim}{\longrightarrow}(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}),P)$, where τ is the canonical filtration.

(0.0.0.21): If S_0 is the spectrum of a perfect field κ of characteristic p and if $S = \mathrm{Spec}(W_n(\kappa))$ $(n > 0)$, then $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ is canonically isomorphic to the filtered complex $(W_n\Omega^{\bullet}_X(\log D), P) := (W_n\Omega^{\bullet}_X(\log D), \{P_kW_n\Omega^{\bullet}_X(\log D)\}_{k\in\mathbb{Z}})$ in [65].

Thus we obtain the following translation which let us recall Grothendieck's project to unify algebra, geometry and analysis ([37]):

(0.0.0.22)

$/\mathbb{C}$	crystal
U_{an}, $(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}$ $((\widetilde{X_{\mathrm{an}}, D_{\mathrm{an}}}))^{\log}_{\mathrm{et}}$ ([51])	$((\widetilde{X, D)/S})^{\log}_{\mathrm{crys}}$
X_{an}, $\widetilde{X_{\mathrm{an}}}$	$\widetilde{(X/S)}_{\mathrm{crys}}$
$j_{\mathrm{an}} \colon U_{\mathrm{an}} \overset{\subset}{\longrightarrow} X_{\mathrm{an}}$ $\epsilon_{\mathrm{top}} \colon (X_{\mathrm{an}}, D_{\mathrm{an}})^{\log} \longrightarrow X_{\mathrm{an}}$ $\epsilon_{\mathrm{an}} \colon ((\widetilde{X_{\mathrm{an}}, D_{\mathrm{an}}}))^{\log}_{\mathrm{et}} \longrightarrow \widetilde{X_{\mathrm{an}}}$	$\epsilon_{(X,D)/S} \colon ((\widetilde{X, D)/S})^{\log}_{\mathrm{crys}} \longrightarrow \widetilde{(X/S)}_{\mathrm{crys}}$
$Rj_{\mathrm{an}*}(\mathbb{Z}) = R\epsilon_{\mathrm{top}*}(\mathbb{Z})$ ([58]), $R\epsilon_{\mathrm{top}*}(\mathbb{Z}/n) = R\epsilon_{\mathrm{an}*}(\mathbb{Z}/n)$ $(n \in \mathbb{Z})$ ([72])	$Q^*_{X/S}R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$
$X_{\mathrm{an}} \longrightarrow X$	$u_{X/S} \colon \widetilde{(X/S)}_{\mathrm{crys}} \longrightarrow \widetilde{X}_{\mathrm{zar}}$
$\mathbb{Z}_{(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}}$ $(\mathbb{Z}/n)_{(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}}$, $(\mathbb{Z}/n)_{((\widetilde{X_{\mathrm{an}}, D_{\mathrm{an}}}))^{\log}_{\mathrm{et}}}$ $(n \in \mathbb{Z})$	$\mathcal{O}_{(X,D)/S}$
$\mathbb{Z}_{X_{\mathrm{an}}}$ $(\mathbb{Z}/n)_{X_{\mathrm{an}}}$ $(n \in \mathbb{Z})$	$\mathcal{O}_{X/S}$
$(\Omega^{\bullet}_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), \tau)$ $= (\Omega^{\bullet}_{X_{\mathrm{an}}/\mathbb{C}}(\log D_{\mathrm{an}}), P)$	$(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), \tau)$ $= (C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$
$(\Omega^{\bullet}_{X/\mathbb{C}}(\log D), P)$	$(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$

Here $(X_{\mathrm{an}}, D_{\mathrm{an}})^{\log}$ is the real blow up of $(X_{\mathrm{an}}, D_{\mathrm{an}})$ defined in [58] and ϵ_{top} is the natural morphism of topological spaces which is denoted by τ in [loc. cit.], and $\widetilde{X_{\mathrm{an}}}$ is the topos defined by the local isomorphisms to X_{an} and ϵ_{an} is the natural morphism forgetting the log structure.

To construct $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$ and $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$, we use local admissible immersions of (X, D) over S, which are local exact closed immersions. On the other hand, in the case where S_0 is the spectrum of a perfect field κ of characteristic $p > 0$ and where $S = \mathrm{Spec}(W_n(\kappa))$, Mokrane has used local lifts of (X, D) over S in [64] and [65] in order to construct the filtered log de Rham-Witt complex $(W_n\Omega^{\bullet}_X(\log D), P)$. Our guiding principle

is: if we can do something by using local lifts, we can do something analogous and more by using admissible immersions. Since a standard exactification of the product of two local lifts of (X, D) is not a local lift of (X, D) at all, the notion "local lift" is not flexible for the construction of the spectral sequence (0.0.0.8). Moreover, because we do not take a local lift of (X, D), we can give a simple proof of the independence of $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$ and $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ of the choice of the open covering of (X, D) and that of the admissible immersion of each open log scheme: we do not need a concrete (slightly) laboring key calculation in [47]; in a future paper we shall develop analogous theory for a family of simple normal crossing log varieties over log points, and we shall show that a concrete key calculation in [48] and [64] is not necessary. Furthermore, we come to know that the filtered log de Rham-Witt complex of an open variety is not a necessary ingredient for the construction of (0.0.0.8) (in the special case $S = \mathrm{Spec}(W_n(\kappa))$). We are sure that it is natural to capture something producing (0.0.0.8) as an object in a filtered derived category; to capture it as a real filtered complex is not flexible. However, because it is anyway possible to capture it as a real filtered complex in the case above, it gives us something deep in a special case. Indeed, in [70], we use the simplicial version of the filtered log de Rham-Witt complex above for the proof of a variant of the Serre-Grothendieck formula on the virtual Betti numbers of a separated scheme of finite type over κ, which has been conjectured in [37].

From the filtered object $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$, we immediately obtain the spectral sequence (0.0.0.8) by (0.0.0.19). If one wishes to obtain only (0.0.0.8), the object $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$ is enough; however, it is an important fact that something producing (0.0.0.8) exists not only in $\mathrm{D}^+\mathrm{F}(f^{-1}(\mathcal{O}_S))$ but also in a "higher stage" $\mathrm{D}^+\mathrm{F}(Q^*_{X/S}(\mathcal{O}_{X/S}))$.

By (0.0.0.17), it is well-known that the canonical filtration τ and the preweight filtration P on $C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S})$ do not coincide in general; however, impressively, τ and P on $C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S})$ coincide ((0.0.0.20)). The equality (0.0.0.20) follows from the following *p-adic purity*

$$\begin{aligned}(0.0.0.23)\qquad & Q^*_{X/S}R^k\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) \\ &= Q^*_{X/S}a^{(k)}_{\mathrm{crys}*}(\mathcal{O}_{D^{(k)}/S}\otimes_{\mathbb{Z}}\varpi^{(k)}_{\mathrm{crys}}(D/S))(-k)\quad (k\in\mathbb{N}),\end{aligned}$$

which will be proved in §2.7. The reason why we obtain the equality (0.0.0.20) is that $(C_{\mathrm{Rcrys}}(\mathcal{O}_{(X,D)/S}), P)$ exists in the world of the classical restricted crystalline topos; we can consider divided powers "$a^{[n]} = a^n/n!$" ($n \in \mathbb{N}$) in the restricted crystalline topos and we can use a Poincaré lemma in it. By (0.0.0.23) and the Poincaré lemma for $R\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$ which will be proved in the text, we obtain (0.0.0.13) and then an important property of $(C_{\mathrm{zar}}(\mathcal{O}_{(X,D)/S}), P)$: it is functorial, that is, for another smooth scheme X' with a relative simple normal crossing divisor D' over S_0 and for a morphism $g\colon (X, D) \longrightarrow (X', D')$ of log schemes over S_0 in the sense of Fontaine-Illusie-

Kato, we have a natural morphism

$$g^*\colon (C_{\rm zar}(\mathcal{O}_{(X',D')/S}),P) \longrightarrow Rg_*(C_{\rm zar}(\mathcal{O}_{(X,D)/S}),P).$$

Finally in this rough explanation of the book, we remark that the following naive p-adic purity

$$R^k\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S}) = a^{(k)}_{{\rm crys}*}(\mathcal{O}_{D^{(k)}/S} \otimes_{\mathbb{Z}} \varpi^{(k)}_{\rm crys}(D/S))(-k) \quad (k \in \mathbb{N})$$

does *not* hold in general, which will be proved in Remark 2.7.11. For this reason, we have to consider the undesirable functor $Q^*_{X/S}$.

Now we outline the contents of this book.

In Chapter 1, we prove preliminary results which we use in later chapters.

From §1.1 to §1.4, we show some facts on filtered modules in a ringed topos which are necessary for later sections. Many key notions and many results are due to P. Berthelot. Especially, the notions *(co)special modules* and *strictly injective resolutions* are due to him. The notion *strictly flat resolutions* is also due to him. The *filtered adjunction formula*, which is also due to him, is a key ingredient for the proof of the filtered base change theorem of $(E_{\rm crys}(\mathcal{O}_{(X,D)/S}),P)$.

In §1.5 and §1.6 we review general facts on diagrams of topoi for this book and future papers.

In Chapter 2, which is the main body of this book, we construct the theory of the weight filtration of the log crystalline cohmologies of families of open smooth varieties.

In §2.1 we give the definition of a relative simple normal crossing divisor and a key notion *admissible immersion* and give a local description of the admissible immersion.

In §2.2 we recall a (log) linearization functor and we calculate the graded pieces of $Q^*_{X/S}L_{X/S}(\Omega^\bullet_{\mathcal{X}/S}(\log \mathcal{D}))$.

In §2.3 we prove a crystalline Poincaré lemma for $R\epsilon_{Y/S*}(E)$ for a morphism $g\colon Y \longrightarrow S_0$ of fine log schemes which can be embedded into a fine log smooth scheme $\mathcal{Y}/S$ whose underlying scheme $\overset{\circ}{\mathcal{Y}}/S$ is also smooth and for a crystal E of $\mathcal{O}_{Y/S}$-modules.

In §2.4 we construct $(C_{\rm Rcrys}(\mathcal{O}_{(X,D)/S}),P)$ and $(C_{\rm zar}(\mathcal{O}_{(X,D)/S}),P)$ for an open covering of X and an admissible immersion of each open log subscheme of (X,D). In §2.5 we prove the independence of $(C_{\rm Rcrys}(\mathcal{O}_{(X,D)/S}),P)$ and $(C_{\rm zar}(\mathcal{O}_{(X,D)/S}),P)$ of the choice of the open covering of X and that of the admissible immersion of each open log subscheme of (X,D).

In §2.6 we prove (0.0.0.18) and (0.0.0.19). In §2.7 we calculate the restriction of the vanishing cycle sheaves $Q^*_{X/S}R^k\epsilon_{(X,D)/S*}(\mathcal{O}_{(X,D)/S})$ $(k \in \mathbb{Z})$ and we prove the p-adic purity (0.0.0.23). Then we prove (0.0.0.13) and (0.0.0.20); as a corollary, we immediately see that $(C_{\rm Rcrys}(\mathcal{O}_{(X,D)/S}),P)$ and

$(C_{\rm zar}(\mathcal{O}_{(X,D)/S}), P)$ are independent of the choice of the decomposition of D by smooth components of D.

In §2.8 we give the description of the boundary morphism of the E_1-terms of the spectral sequence (0.0.0.8).

In §2.9 we prove the functoriality of $(C_{\rm zar}(\mathcal{O}_{(X,D)/S}), P)$.

In §2.10 we prove the filtered base change theorem of $(E_{\rm crys}(\mathcal{O}_{(X,D)/S}), P)$, and we prove the filtered Künneth formula of $(E_{\rm crys}(\mathcal{O}_{(X,D)/S}), P)$. As in [3] and [11], we have some important corollaries of the filtered base change theorem.

In §2.11 we develop the analogous theory for a log crystalline cohomology with compact support, and, especially, we obtain the *preweight spectral sequence* of a log crystalline cohomology with compact support.

In §2.12 we prove that, if S_0 is the spectrum of a perfect field κ of characteristic $p > 0$ and if $S = \mathrm{Spec}(W_n(\kappa))$, then $(C_{\rm zar}(\mathcal{O}_{(X,D)/S}), P)$ is canonically isomorphic to $(W_n\Omega^{\bullet}_X(\log D), P)$.

In §2.13 we prove the convergence of the weight filtration as a corollary of the filtered base change theorem in §2.10. This is a filtered version of the convergence of a (log) crystalline cohomology in [74], [76] and [29].

In §2.14 we give a specialization argument of Deligne-Illusie ([49]) in our situation.

In §2.15 we prove the E_2-degeneration of (0.0.0.8) modulo torsion in the case where $S_0 := \mathrm{Spec}(\kappa)$ and $S = \mathrm{Spf}(W(\kappa))$. The proof in this book is another, more natural proof than the proof in [68]; In §2.16 we prove that a filtered log Berthelot-Ogus isomorphism is strictly compatible with the weight filtration. In §2.17 we generalize the E_2-degeneration of (0.0.0.8) modulo torsion to the case where the base scheme S is a p-adic formal V-scheme in the sense of [74], by using the results in the previous two sections. Using this generalized result, we reprove the convergence of the weight filtration by reducing it to a result in [74].

In §2.18 we prove the strict compatibility of the weight filtration with respect to the induced morphism of log schemes by using the specialization argument in §2.14 and the convergence of the weight filtration.

In §2.19 we prove the strict compatibility of the weight filtration with the Poincaré duality.

In §2.20 we give a remark on the corresponding l-adic weight spectral sequence.

Following the suggestion of one of the referees, we state some results obtained in [70] in Chapter 3 to answer natural questions arising from results in Chapter 2.

For a (separated) scheme of finite type U over $\mathbb{C}$, P. Deligne has endowed $H^h(U_{\rm an}, \mathbb{Q})$ $(h \in \mathbb{Z})$ with the mixed Hodge structure in [24] by using results in [23]. In [70] the first-named author has succeeded in defining the weight filtration on the rigid cohomology of a separated scheme U of finite type over a perfect field κ of characteristic $p > 0$ by using de Jong's alteration theorem

([28]), Tsuzuki's proper descent ([87]), Shiho's comparison theorems ([82]) and results up to Chapter 2.

In §3.1 we give preliminaries for later sections: the single complex of a complex in a multisimplicial topos in [24], the diagonal filtration in [loc. cit.], and the key functor Γ in [19], [87] and [88].

In §3.2 we give a reformulation of Tsuzuki's proper descent on rigid cohomology.

Let $U \overset{\subset}{\longrightarrow} \overline{U}$ be an open immersion into a proper scheme over κ. In §3.3 we state comparison theorems between the rigid cohomology of U, the log convergent cohomology of a certain split proper hypercovering of $(U, \overline{U})$ and the log crystalline cohomology of the proper hypercovering. As an application of the comparison theorems, we can endow the rigid cohomology with closed support with the weight filtration (§3.4). As another application, we can calculate the slope filtration on the rigid cohomology by the log Hodge-Witt sheaves of the proper hypercovering (§3.5). In §3.6 we state the existence theorem of the weight filtration on the rigid cohomology with compact support.

Finally we add the Appendix: we give some results on relative simple normal crossing divisors which are used in the text or related to it.

In a future paper we shall construct an analogous theory for a family of semistable varieties, which is a generalization of [64] and [68].

Notations. (1) For a log scheme Y, $\overset{\circ}{Y}$ denotes the underlying scheme of Y. For a morphism $f\colon Y \longrightarrow Z$ of log schemes, $\overset{\circ}{f}$ denotes the underlying morphism $\overset{\circ}{Y} \longrightarrow \overset{\circ}{Z}$ of schemes.

(2) (S)NCD=(simple) normal crossing divisor.

Conventions. We assume that the log structures on (formal) schemes are defined on Zariski site unless otherwise stated.

Also, we make the following conventions about signs. Let $\mathcal{A}$ be an additive exact category.

(1) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer n, $(E^{\bullet+n}, d^{\bullet+n})$ or $(E^\bullet\{n\}, d^\bullet\{n\})$ denotes the following complex:

$$\cdots \longrightarrow \underset{q-1}{E^{q-1+n}} \overset{d^{q-1+n}}{\longrightarrow} \underset{q}{E^{q+n}} \overset{d^{q+n}}{\longrightarrow} \underset{q+1}{E^{q+1+n}} \overset{d^{q+1+n}}{\longrightarrow} \cdots .$$

Here the numbers under the objects above in $\mathcal{A}$ mean the degrees.

For a morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ of complexes of objects of $\mathcal{A}$, $f\{n\}$ denotes a natural morphism $(E^\bullet\{n\}, d^\bullet_E\{n\}) \longrightarrow (F^\bullet\{n\}, d^\bullet_F\{n\})$ induced by f. A morphism $f\colon (E^\bullet, d^\bullet_E) \longrightarrow (F^\bullet, d^\bullet_F)$ in the derived category $\mathrm{D}^\star(\mathcal{A})$ ($\star = \mathrm{b}, +, -,$ nothing) of the complexes of objects in $\mathcal{A}$ naturally induces a morphism $f\{n\}\colon (E^\bullet\{n\}, d^\bullet_E\{n\}) \longrightarrow (F^\bullet\{n\}, d^\bullet_F\{n\})$ in $\mathrm{D}^\star(\mathcal{A})$.

(2) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$ and for an integer n, $(E^\bullet[n], d^\bullet[n])$ denotes the following complex as usual: $(E^\bullet[n])^q := E^{q+n}$ with boundary morphism $d^\bullet[n] = (-1)^n d^{\bullet+n}$.

For a morphism $f\colon (E^\bullet, d_E^\bullet) \longrightarrow (F^\bullet, d_F^\bullet)$ of complexes of objects of $\mathcal{A}$, $f[n]$ denotes a natural morphism $(E^\bullet[n], d_E^\bullet[n]) \longrightarrow (F^\bullet[n], d_F^\bullet[n])$ induced by f without change of signs. This operation is well-defined in the derived category as in (1).

(3) ([10, 0.3.2], [18, (1.3.2)]) For a short exact sequence

$$0 \longrightarrow (E^\bullet, d_E^\bullet) \xrightarrow{f} (F^\bullet, d_F^\bullet) \xrightarrow{g} (G^\bullet, d_G^\bullet) \longrightarrow 0$$

of bounded below complexes of objects in $\mathcal{A}$, let $(E^\bullet[1], d_E^\bullet[1]) \oplus (F^\bullet, d_F^\bullet)$ be the mapping cone of f. We fix an isomorphism "$(E^\bullet[1], d_E^\bullet[1]) \oplus (F^\bullet, d_F^\bullet) \ni (x, y) \longmapsto g(y) \in (G^\bullet, d_G^\bullet)$" in the derived category $\mathrm{D}^+(\mathcal{A})$.

(4) ([10, 0.3.2], [18, (1.3.3)]) Under the situation (3), the boundary morphism $(G^\bullet, d_G^\bullet) \longrightarrow (E^\bullet[1], d_E^\bullet[1])$ in $\mathrm{D}^+(\mathcal{A})$ is the following composite morphism

$$(G^\bullet, d_G^\bullet) \xleftarrow{\sim} (E^\bullet[1], d_E^\bullet[1]) \oplus (F^\bullet, d_F^\bullet) \xrightarrow{\text{proj.}} (E^\bullet[1], d_E^\bullet[1]) \xrightarrow{(-1)\times} (E^\bullet[1], d_E^\bullet[1]).$$

(5) Let $\mathcal{A}$ be an abelian category with enough injectives. Let $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor of abelian categories. Then, in the situation (3), the boundary morphism $\partial\colon R^q\mathcal{F}((G^\bullet, d_G^\bullet)) \longrightarrow R^{q+1}\mathcal{F}((E^\bullet, d_E^\bullet))$ of cohomologies is, by definition, the induced morphism by the morphism $(G^\bullet, d_G^\bullet) \longrightarrow (E^\bullet[1], d_E^\bullet[1])$ in (4). By taking injective resolutions $(I^\bullet, d_I^\bullet)$, $(J^\bullet, d_J^\bullet)$ and $(K^\bullet, d_K^\bullet)$ of $(E^\bullet, d_E^\bullet)$, $(F^\bullet, d_F^\bullet)$ and $(G^\bullet, d_G^\bullet)$, respectively, which fit into the following commutative diagram

(0.0.0.24)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & (I^\bullet, d_I^\bullet) & \longrightarrow & (J^\bullet, d_J^\bullet) & \longrightarrow & (K^\bullet, d_K^\bullet) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (E^\bullet, d_E^\bullet) & \longrightarrow & (F^\bullet, d_F^\bullet) & \longrightarrow & (G^\bullet, d_G^\bullet) & \longrightarrow & 0
\end{array}$$

of complexes of objects in $\mathcal{A}$ such that the upper horizontal sequence is exact, it is easy to check that the boundary morphism ∂ above is equal to the usual boundary morphism obtained from the upper short exact sequence of (0.0.0.24). (For a short exact sequence in (3), the existence of the commutative diagram (0.0.0.24) has been proved in, e.g., (1.1.7) below, as a very special case.)

(6) For a complex $(E^\bullet, d^\bullet)$ of objects in $\mathcal{A}$, the identity $\mathrm{id}\colon E^q \longrightarrow E^q$ $(\forall q \in \mathbb{Z})$ induces an isomorphism $\mathcal{H}^q((E^\bullet, -d^\bullet)) \xrightarrow{\sim} \mathcal{H}^q((E^\bullet, d^\bullet))$ $(\forall q \in \mathbb{Z})$ of cohomologies. We sometimes use this convention.

(7) We often denote a complex $(E^\bullet, d^\bullet)$ simply by $(E^\bullet, d)$ or $E^\bullet$ as usual when there is no risk of confusion.

(8) Let $r \geq 2$ be a positive integer. As usual, an r-uple complex of objects in $\mathcal{A}$ is, by definition, a pair $(E^{\bullet\cdots\bullet}, \{d_i\}_{i=1}^r)$ such that $E^{m_1\cdots m_r}$ $(m_i \in \mathbb{Z})$ is an object of $\mathcal{A}$ with morphisms $d_i \colon E^{\bullet\cdots\bullet, m_i, \bullet\cdots\bullet} \longrightarrow E^{\bullet\cdots\bullet, m_i+1, \bullet\cdots\bullet}$ satisfying the following relations $d_i^2 = 0$ and $d_i d_j + d_j d_i = 0$ $(i \neq j)$.