Part VI

Free Probability

Free probability is a probability theory for non-commutative variables. In this field, random variables are usually bounded operators on a Hilbert space. The law of a self-adjoint operator T is given as the evaluation $(\langle \zeta, T^n \zeta \rangle)_{n \geq 0}$ of its moments in the direction of a fixed vector ζ of this Hilbert space. Large $N \times N$ matrices can be seen to fit in this framework as bounded operators on the Hilbert space $\mathbb{C}^N$ equipped for instance with the Euclidean scalar product. We will see in fact that free probability is the right framework to consider random matrices as their size goes to infinity.

For the sake of completeness, but actually not needed for our purpose, we shall recall some notions of operator algebras. We shall then describe free probability as a probability theory on non-commutative functionals (a point of view that forgets the space of realizations of the laws) equipped with the notion of freeness that generalizes the idea of independence to this non-commutative setting. We will then focus on large random matrices and show that their asymptotics are related with freeness. In particular, independent Wigner's matrices converge to free semi-circular operators and the Hermitian Brownian motion converges to the free Brownian motion. Conversely, large random matrices can be seen as an approximation to a large class of (and maybe all) operators. In particular, ideas from classical probability, once applied to large random matrices, can be imported to operator algebra theory via such an approximating scheme. In this part, we shall emphasize the uses of stochastic dynamics, as applied to the Hermitian and the free Brownian motions, to obtain large deviations estimates and study free entropies.