Part V

Stochastic Calculus

We shall now study the Hermitian Brownian motion. It is a matrix-valued process $(H_t^N)_{t\geq 0}$ constructed as Gaussian Wigner matrices but with Brownian motion entries instead of Gaussian entries. We shall describe below the symmetric and the Hermitian Brownian motions, leaving the generalization to the symplectic Brownian motions as exercises. We define the symmetric (resp. Hermitian) Brownian motion $H^{N,\beta}$ for $\beta = 1$ (resp. $\beta = 2$) as a process with values in the set of $N \times N$ symmetric (resp. Hermitian) matrices with entries $\left\{H_{i,j}^{N,\beta}(t), t \geq 0, i \leq j\right\}$ constructed via independent real-valued Brownian motions $(B_{i,j}, \tilde{B}_{i,j}, 1 \leq i \leq j \leq N)$ by

$$H_{k,l}^{N,\beta}(t) = \begin{cases} \frac{1}{\sqrt{\beta N}}(B_{k,l}(t) + i(\beta-1)\tilde{B}_{k,l}(t)), & \text{if } k < l \\ \frac{\sqrt{2}}{\sqrt{\beta N}} B_{l,l}(t), & \text{if } k = l. \end{cases} \tag{V.1}$$

Considering the matrix-valued processes, and the associated dynamics, has the advantage to allow us not only to consider one Gaussian Wigner matrix $X^N = H^{N,\beta}(1)$ but also, if $X^N(0)$ is some Hermitian Wigner matrix, the sum $X^N(1) = H^{N,\beta}(1) + X^N(0)$ seen as the matrix at time one of the matrix-valued process $X^N(t) = H^{N,\beta}(t) + X^N(0)$. Studying the evolution of the eigenvalues of $X^N(t)$ allows us to prove the law of large numbers for the spectral measure of $X^N(1)$ (see Lemma 12.5) as well as large deviation principles (see Theorem 13.1). The latter large deviations estimates result in the asymptotics for the spherical or Itzykson–Zuber–Harich–Chandra integrals (see Theorem 14.1) that in turn will give us the value of free energies for diverse two matrices matrix models (see Theorem 15.1) as well as estimates on Schur functions (see Corollary 14.2).