Part IV

Eigenvalues of Gaussian Wigner Matrices and Large Deviations

In this part, we consider the case where the entries of the matrix $\mathbf{X}^{N,\beta}$ are the so-called Gaussian ensembles. Moreover, since the results depend upon the fact that the entries are real or complex, we now show the difference in the notations. We consider $N \times N$ self-adjoint random matrices with entries

$$X_{kl}^{N,\beta} = \frac{\sum_{i=1}^{\beta} g_{kl}^i e_\beta^i}{\sqrt{\beta N}}, \quad 1 \le k < l \le N, \quad X_{kk}^{N,\beta} = \sqrt{\frac{2}{\beta N}} g_{kk} e_\beta^1, \quad 1 \le k \le N$$

where $(e_\beta^i)_{1 \le i \le \beta}$ is a basis of $\mathbb{R}^\beta$, that is $e_1^1 = 1$, $e_2^1 = 1, e_2^2 = i$. This definition can be extended to the case $\beta = 4$, named the Gaussian symplectic ensemble, when N is even by choosing $\mathbf{X}^{N,\beta} = \left(X_{ij}^{N,\beta}\right)_{1 \le i,j \le \frac{N}{2}}$ with $X_{kl}^{N,\beta}$ a 2×2 matrix defined as above but with $(e_\beta^k)_{1 \le k \le 4}$ the Pauli matrices

$$e_4^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_4^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_4^3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_4^4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

$(g_{kl}^i, k \le l, 1 \le i \le \beta)$ are independent equidistributed centered Gaussian variables with variance 1. $(\mathbf{X}^{N,2}, N \in \mathbb{N})$ is commonly referred to as the Gaussian Unitary Ensemble (**GUE**), $(\mathbf{X}^{N,1}, N \in \mathbb{N})$ as the Gaussian Orthogonal Ensemble (**GOE**) and $(\mathbf{X}^{N,4}, N \in \mathbb{N})$ as the Gaussian Symplectic Ensemble (**GSE**) since they can be characterized by the fact that their laws are invariant under the action of the unitary, orthogonal and symplectic group respectively (see [153]). We denote by $P_N^{(\beta)}$ the law of $\mathbf{X}^{N,\beta}$.

The main advantage of the Gaussian ensembles is that the law of the eigenvalues of these matrices is explicit and rather simple. Namely, we now discuss the following lemma.

Lemma IV.1. *Let* $\mathbf{X} \in \mathcal{H}_N^{(\beta)}$ *be random with law* $P_N^{(\beta)}$*. The joint distribution of the eigenvalues* $\lambda_1(X) \le \cdots \le \lambda_N(X)$*, has density proportional to*

$$1_{x_1 \leq \cdots \leq x_N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^{N} e^{-\beta x_i^2/4}. \tag{IV.12}$$

We shall prove this lemma later, when studying Dyson's Brownian motion, see Corollary 12.4. Let us, however, emphasize the ideas behind a direct proof in the case $\beta = 1$. It is simply to write the decomposition $X = UDU^*$, with the eigenvalues matrix D that is diagonal and with real entries, and with the eigenvectors matrix U (that is unitary). Suppose this map was a bijection (which it is not, at least at the matrices X that do not possess all distinct eigenvalues) and that one can parametrize the eigenvectors by $\beta N(N-1)/2$ parameters in a smooth way (which one cannot in general). Then, it is easy to deduce from the formula $X = UDU^*$ that the Jacobian of this change of variables depends polynomially on the entries of D and is of degree $\beta N(N-1)/2$ in these variables. Since the bijection must break down when $D_{ii} = D_{jj}$ for some $i \neq j$, the Jacobian must vanish on that set. When $\beta = 1$, this imposes that the polynomial must be proportional to $\prod_{1 \leq i < j \leq N}(x_i - x_j)$. Further degree and symmetry considerations allow us to generalize this to $\beta = 2$. We refer the reader to [6] for a full proof, that shows that the set of matrices for which the above manipulations are not permitted has Lebesgue measure zero.