Part III

Matrix Models

In this part, we study matrix models, that is, the laws of interacting Hermitian matrices of the form

$$d\mu_V^{N,2}(\mathbf{A}_1,\dots,\mathbf{A}_m) := \frac{1}{Z_V^N} e^{-N\mathrm{Tr}(V(\mathbf{A}_1,\dots,\mathbf{A}_m))} d\mu^{N,2}(\mathbf{A}_1)\cdots d\mu^{N,2}(\mathbf{A}_m)$$

where Z_V^N is the normalizing constant given by the matrix integral

$$Z_V^N = \int e^{-N\mathrm{Tr}(V(\mathbf{A}_1,\dots,\mathbf{A}_m))} d\mu^{N,2}(\mathbf{A}_1)\cdots d\mu^{N,2}(\mathbf{A}_m)$$

and V is a polynomial in m non-commutative variables:

$$V(X_1,\dots,X_m) = \sum_{i=1}^{n} t_i q_i(X_1,\dots,X_m)$$

with q_i non-commutative monomials:

$$q_i(X_1,\dots,X_m) = X_{j_1^i}\cdots X_{j_{r_i}^i}$$

for some $j_l^k \in \{1,\dots,m\}$, $r_i \geq 1$. Moreover, $d\mu^{N,2}(\mathbf{A})$ denotes the standard law of the **GUE**, i.e., under $d\mu^{N,2}(\mathbf{A})$, $\mathbf{A}$ is an $N\times N$ Hermitian matrix such that

$$A(k,l) = \bar{A}(l,k) = \frac{g_{kl} + i\tilde{g}_{kl}}{\sqrt{2N}},\ k<l, \quad A(k,k) = \frac{g_{kk}}{\sqrt{N}}$$

with independent centered standard Gaussian variables $(g_{kl},\tilde{g}_{kl})_{k\leq l}$. In other words

$$d\mu^{N,2}(\mathbf{A}) = Z_N^{-1} 1_{\mathbf{A}\in\mathcal{H}_N^{(2)}} e^{-\frac{N}{2}\mathrm{Tr}(\mathbf{A}^2)} \prod_{1\leq i\leq j\leq N} d\Re(A(i,j)) \prod_{1\leq i<j\leq N} d\Im(A(i,j)).$$

Since we restrict ourselves to Hermitian matrices in this part, we shall drop the subscript $\beta=2$ and write for short $\mu^N = \mu^{N,2}$.

Let us define by $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ the set of polynomials in m non-commutative variables and, for $P \in \mathbb{C}\langle X_1, \ldots, X_m\rangle$,

$$\mathbf{L}(P) := \mathbf{L}_{\mathbf{A}_1,\ldots,\mathbf{A}_m}(P) = \frac{1}{N}\mathrm{Tr}\left(P(\mathbf{A}_1, \ldots, \mathbf{A}_m)\right).$$

When V vanishes, we have seen in Chapter 3 that for any polynomial function P, $\mathbf{L}(P)$ converges as N goes to infinity. Moreover the limit $\sigma^m(P)$ is such that if P is a monomial, $\sigma^m(P)$ is the number of non-crossing pair partitions of a set of points with m colors, or equivalently the number of planar maps with one star of type P. In this part, we shall generalize such a type of result to the case where V does not vanish but is “small” and “nice” in a sense that we shall precise.

This part is motivated by a work of Brézin, Parisi, Itzykson and Zuber [50] and large developments that occurred thereafter in theoretical physics [78]. They specialized an idea of ’t Hooft [187] to show that if $V = \sum_{i=1}^n t_i q_i$ with fixed monomials q_i of m non-commutative variables, and if we see $Z_V^N = Z_{\mathbf{t}}^N$ as a function of $\mathbf{t} = (t_1, \ldots, t_n)$,

$$\log Z_{\mathbf{t}}^N := \sum_{g\geq 0} N^{2-2g} F_g(\mathbf{t}), \tag{III.15}$$

where

$$F_g(\mathbf{t}) := \sum_{k_1,\ldots,k_n \in \mathbb{N}^k} \prod_{i=1}^k \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_g((q_i, k_i)_{1\leq i\leq k})$$

is a generating function of integer numbers $\mathcal{M}_g((q_i, k_i)_{1\leq i\leq k})$ that count certain graphs called maps. A map is a connected oriented graph that is embedded into a surface. Its genus g is by definition the genus of a surface in which it can be embedded in such a way that edges do not cross and the faces of the graph (that are defined by following the boundary of the graph) are homeomorphic to a disk. The vertices of the maps we shall consider will have the structure of a star, that is a vertex with colored edges embedded into a surface (that is an order on the colored edges is specified). More precisely, a star of type q, for some monomial $q = X_{\ell_1}\cdots X_{\ell_k}$, is a vertex with degree $\deg(q)$ and oriented colored half-edges with one marked half edge of color ℓ_1, the second of color ℓ_2, etc., until the last one of color ℓ_k. $\mathcal{M}_g((q_i, k_i)_{1\leq i\leq k})$ is then the number of maps with k_i stars of type q_i, $1 \leq i \leq n$.

Adding to V a term $t\,q$ for some monomial q and identifying the first-order derivative with respect to t at $t = 0$ we derive from (III.15)

$$\int \mathbf{L}(q)d\mu_V^N = \sum_{g\geq 0} N^{-2g} \sum_{k_1,\ldots,k_n \in \mathbb{N}^k} \prod_{i=1}^k \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_g((q_i, k_i)_{1\leq i\leq k}, (q, 1)). \tag{III.16}$$

The equalities (III.15) and (III.16) derived in [50] are only formal, i.e., mean that all the derivatives on both sides of the equality coincide at $\mathbf{t} = 0$. They

can thus be deduced from the Wick formula (which gives the expression of arbitrary moments of Gaussian variables) or equivalently by the use of Feynman diagrams.

Even though topological expansions such as (III.15) and (III.16) were first introduced by 't Hooft in the course of computing the integrals, the natural reverse question of computing the numbers $\mathcal{M}_g((q_i, k_i)_{1\le i\le k})$ by studying the associated integrals over matrices encountered a large success in theoretical physics (see, e.g., the review papers [70, 78]). In the course of doing so, one would like for instance to compute $\lim_{N\to\infty} N^{-2}\log Z_{\mathbf{t}}^N$ and claim that this limit is equal to $F_0(\mathbf{t})$. There is here the belief that one can interchange derivatives and limit, a claim that needs to be justified.

We shall indeed prove that the formal limit can be strenghtened into a large N expansion in the sense that

$$\frac{1}{N^2}\log Z_{\mathbf{t}}^N = F_0(\mathbf{t}) + \frac{1}{N^2}F_1(\mathbf{t}) + o(N^{-2})$$

where $N^2 \times o(N^{-2})$ goes to zero as N goes to infinity. This asymptotic expansion holds when V is small and satisfies some convexity hypothesis (which insures that the partition function Z_V^N is finite and the support of the limiting spectral measures of $\mathbf{A}_i$, $1 \le i \le m$, under μ_V^N is connected, see [106]).

This part summarizes results from [104] and [105]. The full expansion (i.e., higher-order corrections) was obtained by E. Maurel Segala [148] in the multi-matrix setting. Such expansion in the one matrix case was already derived on a physical level of rigor in [4] and then made rigorous in [2, 86]. However, in the case of one matrix, techniques based on orthogonal polynomials can be used. In the multi-matrix case this approach fails in general (or at least has not yet been extended). [104, 105, 148] take a completely different route based on the free probability setting of limiting tracial states and of the so-called Master loop or Schwinger–Dyson equations.

We start this part by introducing the combinatorial objects we shall consider and their relations with non-commutative polynomials. Then, we prove the formal expansion of Brézin, Itzykson, Parisi and Zuber. The next two chapters consider the asymptotic expansion; we first obtain the convergence of the free energy towards the expected generating function for the enumeration of planar maps, and then study the first order correction to this limit, showing it is related with the enumeration of maps with genus one.

The techniques we shall present here have the advantage to be robust. We use them here to study partition functions of Hermitian matrices, but they can be generalized to orthogonal or symplectic matrices (in a work in progress of E. Maurel Segala) or to matrices following the Haar measure on the unitary group [66]. The last extension is particularly interesting since then Gaussian calculus and Feynman diagram techniques fail (since unitary matrices have no Gaussian entries) so that the diagrammatic representation of the limit is not straightforward even on a formal level (see [65] for a formal expansion with no diagrammatic interpretation).