Part II

Wigner Matrices and Concentration Inequalities

In the last twenty years, concentration inequalities have developed into a very powerful tool in probability theory. They provide a general framework to control the probability of deviations of smooth functions of random variables from their mean or their median. We begin this section by providing some general framework where concentration inequalities are known to hold. We first consider the case where the underlying measure satisfies a log-Sobolev inequality; we show how to prove this inequality in a simple situation and then how it implies concentration inequalities. We then review a few other situations where concentration inequalities hold. To apply these techniques to random matrices, we show that certain functions of the eigenvalues of matrices, such as $\int f(x) dL_{\mathbf{A}^N}(x)$ with f Lipschitz, are smooth functions of the entries of the matrix $\mathbf{A}^N$ so that concentration inequalities hold as soon as the joint law of the entries satisfies one of the conditions seen in the first two chapters of this part. Another useful *a priori* control is provided by Brascamp–Lieb inequalities; we shall apply them to the setting of random matrices at the end of this part.

To motivate the reader, let us state the type of result we want to obtain in this part.

To this end, we introduce some extra notations. Let us recall that if X is a symmetric (resp. Hermitian) matrix and f is a bounded measurable function, $f(X)$ is defined as the matrix with the same eigenvectors than X but with eigenvalues that are the image by f of those of X; namely, if e is an eigenvector of X with eigenvalue λ, $Xe = \lambda e$, $f(X)e := f(\lambda)e$. In terms of the spectral decomposition $X = UDU^*$ with U orthogonal (resp. unitary) and D diagonal real, one has $f(X) = Uf(D)U^*$ with $f(D)_{ii} = f(D_{ii})$. For $M \in \mathbb{N}$, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product on $\mathbb{R}^M$ (resp. $\mathbb{C}^M$), $\langle x, y \rangle = \sum_{i=1}^M x_i y_i$ ($\langle x, y \rangle := \sum_{i=1}^M x_i y_i^*$), and by $|| \cdot ||_2$ the associated norm $||x||_2^2 := \langle x, x \rangle$.

Throughout this section, we denote the Lipschitz constant of a function $G : \mathbb{R}^M \to \mathbb{R}$ by

$$|G|_{\mathcal{L}} := \sup_{x \neq y \in \mathbb{R}^M} \frac{|G(x) - G(y)|}{\|x - y\|_2},$$

and call G a *Lipschitz function* if $|G|_{\mathcal{L}} < \infty$.

Lemma II.1. *Let $g : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz with Lipschitz constant $|g|_{\mathcal{L}}$. Then, with $\mathbf{A}^N$ denoting the Hermitian (or symmetric) matrix with entries $(A_{ij}^N)_{1 \le i,j \le N}$, the map $\{A_{ij}^N\}_{1 \le i \le j \le N} \mapsto \mathrm{Tr}(g(\mathbf{A}^N))$ is a Lipschitz function with constant $\sqrt{N}|g|_{\mathcal{L}}$. Therefore, if the joint law of $(A_{ij}^N)_{1 \le i \le j \le N}$ is "good", there exists $\alpha > 0$, constants $c > 0$ and $C < \infty$ so that for all $N \in \mathbb{N}$*

$$\mathbb{P}\left(\left|\mathrm{Tr}(g(\mathbf{A}^N)) - \mathbb{E}[\mathrm{Tr}(g(\mathbf{A}^N))]\right| > \delta |g|_{\mathcal{L}}\right) \le C e^{-c|\delta|^{\alpha}}.$$

"Good" here means for instance that the law satisfies a log-Sobolev inequality; an example is when the $\{A_{ij}^N\}_{1 \le i \le j \le N}$ are independent Gaussian variables with uniformly bounded covariance (see Theorem 6.6).

The interest of results such as Lemma II.1 is that they provide bounds on deviations that do not depend on the dimension. They can be used to show laws of large numbers (reducing the proof of the almost sure convergence to the prove of the convergence in expectation) or to ease the proof of central limit theorems (indeed, when $\alpha = 2$ in Lemma II.1, $\mathrm{Tr}(g(\mathbf{A}^N)) - \mathbb{E}[\mathrm{Tr}(g(\mathbf{A}^N))]$ has a sub-Gaussian tail, providing tightness arguments for free).

We shall recall below the elements of the theory of concentration we shall need. In fact, we will mostly use concentration inequalities related to log-Sobolev inequalities; we shall therefore provide details on this point and give full proofs. We will then review other classical settings where concentration inequalities are known to apply. Finally, we will apply this theory to random matrices and provide for instance sufficient hypotheses so that Lemma II.1 holds.