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Large Random Matrices: Lectures on Macroscopic Asymptotics

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Preface

These notes include the material from a series of nine lectures given at the Saint-Flour probability summer school in 2006. The two other lecturers that year were Maury Bramson and Steffen Lauritzen.

The topic of these lectures was large random matrices, and more precisely the asymptotics of their macroscopic observables such as the empirical measure of their eigenvalues. The interest in such questions goes back to Wishart and Wigner, in the twenties and fifties respectively. Large random matrices have been since then intensively studied in theoretical physics, in connection with various fields such as QCD, quantum chaos, string theory or quantum gravity.

Since the nineties, several key mathematical results have been obtained and the theory of large random matrices expanded in various directions, in connection with combinatorics, operator algebra theory, number theory, algebraic geometry, integrable systems etc. I felt that the time was right to summarize some of them, namely those which connect with the asymptotics of macroscopic observables, with a particular emphasis on their relation with combinatorics and operator algebra theory.

I wish to thank Jean Picard for organizing the Saint-Flour school and helping me through the preparation of these notes, and the other participants of the school, in particular for their useful comments to improve these notes. I am very grateful to several collaborators with whom I consulted on various points, in particular Greg Anderson, Edouard Maurel Segala, Dima Shlyakhtenko and Ofer Zeitouni.

Lyon, France *Alice Guionnet*
July 2008

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Notation

• $\mathcal{C}_b(\mathbb{R})$ (resp. $\mathcal{C}_b^1(\mathbb{R}^N,\mathbb{R})$) denotes the space of bounded continuous functions on $\mathbb{R}$ (resp. k times continuously differentiable functions from $\mathbb{R}^N$ into $\mathbb{R}$). If f is a real-valued function on a metric space (X,d),

$$\|f\|_\infty = \sup_{x\in X} |f(x)|$$

denotes its supremum norm, whereas we set the Lipschitz norms to be

$$\|f\|_{\mathcal{L}} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)} + \sup_x |f(x)|, \quad |f|_{\mathcal{L}} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}$$

For $x\in\mathbb{R}^N$, and $f\in\mathcal{C}_b^1(\mathbb{R}^N,\mathbb{R})$, we let

$$\|x\|_2 = \left(\sum_{i=1}^N (x_i)^2\right)^{\frac{1}{2}}, \quad \|\nabla f\|_2 = \left(\sum_{i=1}^N (\partial_{x_i} f(x))^2\right)^{\frac{1}{2}}.$$

• $\mathcal{P}(X)$ denotes the set of probability measures on the metric space (X,d). $\mu(f)$ is a shorthand for $\int f(x)d\mu(x)$. We shall call the weak topology on $\mathcal{P}(X)$ the topology so that $\mu\to\mu(f)$ is continuous if f is bounded continuous on (X,d). The moments topology refers to the continuity of $\mu\to\mu(x^k)$ for all $k\in\mathbb{N}$. Even though both topologies coincide if X is compact subset of $\mathbb{R}$, they can be different in general.

• If (X,d) is a metric space, Dudley's distance d_D on $\mathcal{P}(X)$ (which is compatible with the weak topology on $\mathcal{P}(X)$) is given by

$$d_D(\mu,\nu) := \sup_{\|f\|_{\mathcal{L}}\le 1} \left| \int f(x)d\mu(x) - \int f(x)d\nu(x) \right| \tag{0.1}$$

• $\mathcal{M}_N(\mathbb{C})$ (resp. $\mathcal{H}_N^{(1)}$, resp. $\mathcal{H}_N^{(2)}$) denotes the set of $N\times N$ (resp. symmetric, resp. Hermitian) matrices with complex (resp. real, resp. complex) coefficients. $\mathcal{M}_N(\mathbb{C})$ is equipped with the trace Tr:

$$\mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii}.$$

• If A is an $N \times N$ Hermitian matrix, we denote by $(\lambda_k(A))_{1 \le k \le N}$ its eigenvalues.

• For A an $N \times N$ matrix, we define

$$\|A\|_2 = \left(\sum_{i,j=1}^{N} |A_{ij}|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|A\|_\infty = \lim_{n\to\infty} \left(\mathrm{Tr}((AA^*)^n)\right)^{\frac{1}{2n}}.$$

The latter norm also coincides with the spectral radius of A which we denote by $\lambda_{\max}(A)$. 1 or I will denote the identity in $\mathcal{M}_N(\mathbb{C})$ and when no confusion is possible, for any constant c, c denotes $c1$.

• $\mathbb{C}\langle X_1, \ldots, X_m\rangle$ denotes the set of polynomials in m non-commutative indeterminates $(X_1, \ldots, X_m)$, $\mathbb{C}\langle X_1, \ldots, X_m\rangle_{sa}$ the subset of polynomials such that $P = P^*$ for some involution $*$ defined on $\mathbb{C}\langle X_1, \ldots, X_m\rangle$.

• Often, bold symbols will indicate vectors, e.g., $\mathbf{X} = (X_1, \ldots, X_m)$ or matrices e.g., $\mathbf{A} = (A_{ij})_{1 \le i,j \le N}$. The letters $(\mathbf{A}, \mathbf{B})$ in general refer to random matrices, whereas (X, Y, Z), to generic (eventually non-commutative) indeterminates.