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Quasi-Periodic Motions in Families of Dynamical Systems

Order amidst Chaos

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Preface

This book is devoted to the phenomenon of quasi-periodic motion in dynamical systems. Such a motion in the phase space densely fills up an invariant torus. This phenomenon is most familiar from Hamiltonian dynamics. Hamiltonian systems are well known for their use in modelling the dynamics related to frictionless mechanics, including the planetary and lunar motions. In this context the general picture appears to be as follows. On the one hand, Hamiltonian systems occur that are in complete order: these are the integrable systems where all motion is confined to invariant tori. On the other hand, systems exist that are entirely chaotic on each energy level. In between we know systems that, being sufficiently small perturbations of integrable ones, exhibit coexistence of order (invariant tori carrying quasi-periodic dynamics) and chaos (the so called stochastic layers). The Kolmogorov–Arnol′d–Moser (KAM) theory on quasi-periodic motions tells us that the occurrence of such motions is *open* within the class of all Hamiltonian systems: in other words, it is a phenomenon persistent under small Hamiltonian perturbations. Moreover, generally, for any such system the union of quasi-periodic tori in the phase space is a nowhere dense set of positive Lebesgue measure, a so called Cantor family. This fact implies that open classes of Hamiltonian systems exist that are not ergodic.

The main aim of the book is to study the changes in this picture when other classes of systems – or contexts – are considered. Examples of such contexts are the class of reversible systems, of volume preserving systems or the class of all systems, often referred to as "dissipative". In all these cases, we are interested in the occurrence of quasi-periodic motions, or tori, persistent under small perturbations within the class in question. By an application of the KAM theory it turns out that in certain cases, in order to have this persistence, the systems are required to depend on *external* parameters. An example of such a situation is the dissipative class, where quasi-periodic attractors are found. These attracting quasi-periodic tori are isolated in the phase space and they are only persistent when at least one parameter is present. In that case, generally, the set of parameters for which such an attractor occurs has positive Lebesgue measure. Quasi-periodic attractors are well known to be a transient stage in a bifurcation route from order to chaos.

The KAM theory is a powerful instrument for the investigation of this problem in a broad sense, describing the organization of invariant tori as Cantor families of positive Lebesgue (Hausdorff) measure. It yields a unifying approach for all cases, leading to a formulation with a minimal number of parameters. In this book, we discuss various aspects of the KAM theory. However, there are still many problems of the theory outside the scope of the present text. Some of these will be briefly indicated.

We proceed in giving an outline of the text. In introductory Chapter 1 we present a more precise formulation of our main problem illustrating this with many examples. Here we also define the contexts to which we apply our approach throughout. These include two different reversible contexts and, in the Hamiltonian setting next to the isotropic case, also the coisotropic one. The Chapters 2 and 3 form the "kernel" of the book. In Chapter 2 first we formulate the conjugacy or stability theory. Depending on the context, we introduce a suitable number of unfolding parameters which stabilize the systems within their context for small perturbations. This stability only holds on Cantor families of invariant tori with Diophantine frequencies (or KAM tori), the corresponding conjugacies being smooth in the sense of Whitney. This approach first leaves us with families that depend on

a great many parameters, but the discussion continues by systematically reducing the number of parameters to a minimum, where still sufficiently many tori are left, in the sense of Hausdorff measure. Main tool here is the Diophantine approximation lemma in the form close to that of V.I. Bakhtin. Chapter 3 subsequently discusses the theory on the continuation of analytic tori due to A.D. Bruno.

Next, in Chapter 4, we discuss the organization of the Cantor families of tori as they occur in our various contexts, including estimates of the appropriate Hausdorff measure. We also present some considerations regarding the dynamics in the "resonance zones", i.e., in the complement of the KAM tori (including Nekhoroshev estimates on solutions near those tori).

Chapter 5 presents conclusive remarks on the subject. Correspondences and differences between the cases of vector fields and diffeomorphisms are discussed. Also we show that, generally, the KAM tori accumulate very much on each other, which can be concisely formulated in terms of Lebesgue density points.

Chapter 6 consists of appendices. In the first of these, the stability theorem is proven in one of its simplest forms. Other appendices fully describe the Bruno theory and the Diophantine approximation lemma.

The style of the book makes it suitable for both experts and beginners regarding the KAM theory. On the one hand, it presents an up to date and therefore quite advanced overview of the theory. On the other hand, it contains an elementary introduction to Whitney differentiability and a complete proof of the simplest stability theorem in this respect. By this and the other details of the appendices, the text is largely self-contained. Also it contains an extended bibliography (which does not, of course, claim to be complete).

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H.W.B., G.B.H. & M.B.S.
Groningen, September 1996

Notation

Although, as a rule, we explain each notation where it first appears in the book, some notations used frequently in the sequel are collected here. Some basic sets are denoted by the "open font" (or "blackboard bold") characters. The symbols Z, Q, R, and C require no comments, $\mathbb{R}_+$ will denote the set of *non-negative* real numbers, $\mathbb{Z}_+$ is the set of *non-negative* integers, and $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$ is the set of positive integers. The symbol $\mathbb{S}^n$ denotes the unit n-dimensional sphere in $\mathbb{R}^{n+1}$, and $\mathbb{T}^n = (\mathbb{S}^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ is the standard n-torus. Also, $\mathbb{RP}^n$ is the n-dimensional real projective space, and $\Pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ denotes the natural projection. Note that each of the spaces $\mathbb{T}^0$ and $\mathbb{RP}^0$ is a point, while $\mathbb{S}^0$ is the disjoint union of two points. Note also that $\mathbb{RP}^1 \simeq \mathbb{S}^1 = \mathbb{T}^1$.

The symbols $\mathcal{O}_n(a)$ will denote a neighborhood of a point $a \in \mathbb{R}^n$. In particular, $\mathcal{O}_n(0)$ is a neighborhood of the origin in $\mathbb{R}^n$. We write $\mathcal{O}(a)$ instead of $\mathcal{O}_1(a)$ for $a \in \mathbb{R}$.

By the angle brackets, we will denote the standard inner product of two vectors, so that

$$\langle a, b\rangle = \textstyle\sum_{i=1}^n a_i b_i \quad \text{for} \quad a \in \mathbb{R}^n, b \in \mathbb{R}^n.$$

The symbols $|a|$ and $\|a\|$ will denote the l_1-norm and l_2-norm (Euclidean norm), respectively, of vector a (unless when stated otherwise):

$$|a| = \textstyle\sum_{i=1}^n |a_i| \quad \text{and} \quad \|a\|^2 = \sum_{i=1}^n |a_i|^2 \quad \text{for} \quad a \in \mathbb{C}^n.$$

For the Landau symbols, we write $O_m(u)$ instead of $O(|u|^m)$ [and $O(u)$ instead of $O_1(u) = O(|u|)$] for $m \in \mathbb{N}$ and any (scalar or vector) independent variable u. By $D_u F$ we will sometimes denote the Jacobi matrix $\partial F/\partial u$. The relation $a := b$ will mean that equality $a = b$ is the definition of quantity a. By $\log u$ we denote $\log_e u$ (which is often designated by $\ln u$ elsewhere). The boundary of manifold or set M is denoted by ∂M. The interior of set M is denoted by $\mathrm{int}(M)$ and the closure of set M by $\mathrm{cl}(M)$ or $\overline{M}$. The symbols $diag(a_1, a_2, \dots, a_n)$ mean the diagonal $n \times n$ matrix with diagonal entries $a_1, a_2, \dots, a_n$. The matrix transposed to A is denoted by A^t. The dot means differentiation with respect to time: $\dot{x} := dx/dt$ and $\ddot{x} := d^2x/dt^2$. The average of a function over $\mathbb{T}^n$ will be sometimes denoted by the square brackets $[\cdot]$. Mark $\square$ means the end of the proof.

The notations $[a, b]$, $[a, b[$, $]a, b]$, and $]a, b[$ for $-\infty \le a < b \le +\infty$ mean respectively the intervals $\{x \ : \ a \le x \le b\}$, $\{x \ : \ a \le x < b\}$, $\{x \ : \ a < x \le b\}$, and $\{x \ : \ a < x < b\}$. For example, $\mathbb{R}_+ = [0, +\infty[$. Given $x \in \mathbb{R}$, the integral part of x is denoted by $\mathrm{Entier}(x) := \max\{m \in \mathbb{Z} \ : \ m \le x\} = \max(\mathbb{Z} \cap]-\infty, x])$. If $\mathrm{Entier}(x) = \ell$ then $\ell \le x < \ell + 1$.

The n-dimensional Hausdorff measure in $\mathbb{R}^N$ for $N \ge n$ (see Federer [114] or Morgan [242]) will be denoted by $meas_n$ (elsewhere usually denoted by $\mathcal{H}^n$). For $N = n$ the measure $meas_n$ coincides with the standard Lebesgue measure $\mathcal{L}^n$ in $\mathbb{R}^n$.

The term "analytic" will always refer to mappings between *real* manifolds (equipped with an analytic structure), whereas holomorphic functions $f : D \to (\mathbb{C}/2\pi\mathbb{Z})^{N_1} \times \mathbb{C}^{N_2}$, $D \subset (\mathbb{C}/2\pi\mathbb{Z})^{n_1} \times \mathbb{C}^{n_2}$, that are real-valued for real arguments will be called "real analytic".

... there are so many ways to deal with formulas.

Donald E. Knuth. *The TEXbook.* Addison-Wesley, 1986.

There are some formulas that can't be handled easily ...

Leslie Lamport. LATEX. *A Document Preparation System.* Addison-Wesley, 1986.

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