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Jin Ma
Jiongmin Yong

Forward-Backward Stochastic Differential Equations and their Applications

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Author

Jin Ma
Department of Mathematics
Purdue University
150 N. University Street
West Lafayette
IN 47906-1395, USA
e-mail: majin@math.purdue.edu

Jiongmin Yong
Department of Mathematics
Fudan University
Shanghai 200433
People's Republic of China
e-mail: jyong@fudan.edu.cn

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To

Yun and Meifen

Preface

This book is intended to give an introduction to the theory of forward-backward stochastic differential equations (FBSDEs, for short) which has received strong attention in recent years because of its interesting structure and its usefulness in various applied fields.

The motivation for studying FBSDEs comes originally from stochastic optimal control theory, that is, the adjoint equation in the Pontryagin-type maximum principle. The earliest version of such an FBSDE was introduced by Bismut [1] in 1973, with a decoupled form, namely, a system of a usual (forward) stochastic differential equation and a (linear) backward stochastic differential equation (BSDE, for short). In 1983, Bensoussan [1] proved the well-posedness of general linear BSDEs by using martingale representation theorem. The first well-posedness result for nonlinear BSDEs was proved in 1990 by Pardoux–Peng [1], while studying the general Pontryagin-type maximum principle for stochastic optimal controls. A little later, Peng [4] discovered that the adapted solution of a BSDE could be used as a probabilistic interpretation of the solutions to some semilinear or quasilinear parabolic partial differential equations (PDE, for short), in the spirit of the well-known Feynman-Kac formula. After this, extensive study of BSDEs was initiated, and potential for its application was found in applied and theoretical areas such as stochastic control, mathematical finance, differential geometry, to mention a few.

The study of (strongly) coupled FBSDEs started in early 90s. In his Ph.D thesis, Antonelli [1] obtained the first result on the solvability of an FBSDE over a "small" time duration. He also constructed a counterexample showing that for coupled FBSDEs, large time duration might lead to non-solvability. In 1993, the present authors started a systematic investigation on the well-posedness of FBSDEs over arbitrary time durations, which has developed into the main body of this book. Today, several methods have been established for solving a (coupled) FBSDE. Among them two are considered effective: the *Four Step Scheme* by Ma–Protter–Yong [1] and the *Method of Continuation* by Hu–Peng [2], and Yong [1]. The former provides the explicit relations among the forward and backward components of the adapted solution via a quasilinear partial differential equation, but requires the non-degeneracy of the forward diffusion and the non-randomness of the coefficients; while the latter relaxed these conditions, but requires essentially the "monotonicity" condition on the coefficients, which is restrictive in a different way.

The theory of FBSDEs have given rise to some other problems that are interesting in their own rights. For example, in order to extend the Four Step Scheme to general random coefficient case, it is not hard to see that one has to replace the quasilinear parabolic PDE there by a quasilinear *backward stochastic* partial differential equation (BSPDE for short), with a

strong degeneracy in the sense of stochastic partial differential equations. Such BSPDEs can be used to generalize the Feynman-Kac formula and even the Black-Scholes option pricing formula to the case when the coefficients of the diffusion are allowed to be random. Other interesting subjects generated by FBSDEs but with independent flavors include FBSDEs with reflecting boundary conditions as well as the numerical methods for FBSDEs. It is worth pointing out that the FBSDEs have also been successfully applied to model and to resolve some interesting problems in mathematical finance, such as problems involving term structure of interest rates (consol rate problem) and hedging contingent claims for large investors, etc.

The book is organized as follows. As an introduction, we present several interesting examples in Chapter 1. After giving the definition of solvability, we study some special FBSDEs that are either non-solvable or easily solvable (e.g., those on small durations). Some comparison results for both BSDE and FBSDE are established at the end of this chapter. In Chapter 2 we content ourselves with the linear FBSDEs. The special structure of the linear equations enables us to treat the problem in a special way, and the solvability is studied thoroughly. The study of general FBSDEs over arbitrary duration starts from Chapter 3. We present virtually the first result regarding the solvability of FBSDE in this generality, by relating the solvability of an FBSDE to the solvability of an optimal stochastic control problem. The notion of *approximate solvability* is also introduced and developed. The idea of this chapter is carried on to the next one, in which the Four Step Scheme is established. Two other different methods leading to the existence and uniqueness of the adapted solution of general FBSDEs are presented in Chapters 6 and 7, while in the latter even reflections are allowed for both forward and backward equations. Chapter 5 deals with a class of linear backward SPDEs, which are closely related to the FBSDEs with random coefficients; Chapter 8 collects some applications of FBSDEs, mainly in mathematical finance, which in a sense is the inspiration for much of our theoretical research. Those readers needing stronger motivation to dig deeply into the subject might actually want to go to this chapter first and then decide which chapter would be the immediate goal to attack. Finally, Chapter 9 provides a numerical method for FBSDEs.

In this book all "headings" (theorem, lemma, definition, corollary, example, etc.) will follow a single sequence of numbers within one chapter (e.g., Theorem 2.1 means the first "heading" in Section 2, possibly followed immediately by Definition 2.2, etc.). When a heading is cited in a different chapter, the chapter number will be indicated. Likewise, the numbering for the equations in the book is of the form, say, (5.4), where 5 is the section number and 4 is the equation number. When an equation in different chapter is cited, the chapter number will precede the section number.

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Jiongmin Yong, Shanghai

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