Part II

Edwin Perkins: Dawson–Watanabe Superprocesses and Measure-valued Diffusions

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Glossary of Notation

$\alpha \sim t$	$\lvert\alpha\rvert/N \le t < (\lvert\alpha\rvert+1)/N = \zeta^\alpha$, i.e., α labels a particle alive at time t
$\lvert A\rvert$	Lebesgue measure of A
A^δ	the set of points less than a distance δ from the set A
$A^g\phi$	$A\phi + g\phi$
$\hat{A}$	weak generator of path-valued Brownian motion–see Lemma V.2.1
$\bar{A}_{\tau,m}$	see Proposition V.2.6
$\vec{A}$	generator of space-time process–see prior to Proposition II.5.8
$\stackrel{bp}{\to}$	bounded pointwise convergence
$b\mathcal{E}$	the set of bounded $\mathcal{E}$-measurable functions
BSMP	a cadlag strong Markov process with $x \mapsto P^x(A)$ Borel measurable for each measurable A in path space
$\mathcal{B}(E)$	the Borel σ-field on E
C	$C(\mathbb{R}^d)$
$C(E)$	continuous E-valued functions on $\mathbb{R}_+$ with the topology of uniform convergence on compacts
$C_b(E)$	bounded continuous E-valued functions on $\mathbb{R}_+$ with the supnorm topology
$C_b^k(\mathbb{R}^d)$	functions in $C_b(\mathbb{R}^d)$ with bounded continuous partials of order k or less
$C_b^\infty(\mathbb{R}^d)$	functions in $C_b(\mathbb{R}^d)$ with bounded continuous partials of any order
$C_K(E)$	continuous functions with compact support on E with the supnorm topology
$C_\ell(E)$	continuous functions on a locally compact E with a finite limit at ∞
$C(g)(A)$	the g-capacity of a set A–see prior to Theorem III.5.2
$\mathcal{C}$	Borel σ-field for $C = C(\mathbb{R}^d)$
$\mathcal{C}_t$	sub-σ-field of $\mathcal{C}$ generated by coordinate maps $y_s, s \le t$
$\stackrel{\mathcal{D}}{\equiv}$	equal in distribution
$D(E)$	the space of cadlag paths from $\mathbb{R}_+$ to E with the Skorokhod J_1 topology
D^s	the set of paths in $D(E)$ which are constant after time s
D_{fd}	smooth functions of finitely many coordinates on $\mathbb{R}_+ \times C$ –see Example V.2.8
$D(n,d)$	space of $\mathbb{R}^{n\times d}$-valued integrands–see after Proposition V.3.1
Δ	cemetary state added to E as a discrete point
$\mathcal{D}(A)$	domain of the weak generator A–see II.2 and Proposition II.2.2
$\mathcal{D}(\vec{A})_T$	domain of weak space-time generator–see prior to Proposition II.5.7
$\mathcal{D}(\hat{A})$	domain of the weak generator for path-valued Brownian motion –see Lemma V.2.1
$\mathcal{D}(\Delta/2)$	domain of the weak generator of Brownian motion–see Example II.2.4
$\mathcal{D}$	the Borel σ-field on the Skorokhod space $D(E)$
$(\mathcal{D}_t)_{t\ge 0}$	the canonical right-continuous filtration on $D(E)$
$e_\phi(W)$	$e^{-W(\phi)}$
$\mathcal{E}$	the Borel σ-field on E
$\mathcal{E}_+$	the non-negative $\mathcal{E}$-measurable functions
$\hat{E}$	$\{(t, y(\cdot \wedge t)) : y \in D(E), t \ge 0\}$
$f_\beta(r)$	r^β if $\beta > 0$, $(\log 1/r)^{-1}$ if $\beta = 0$
$\hat{\mathcal{F}}$	$\mathcal{F} \times \mathcal{B}(C(\mathbb{R}^d))$
$\hat{\mathcal{F}}_t$	$\mathcal{F}_t \times \mathcal{C}_t$
$\hat{\mathcal{F}}_t^*$	the universal completion of $\hat{\mathcal{F}}_t$

$\mathcal{F}_X$	the Borel σ-field on Ω_X
$g_\beta(r)$	$r^{-\beta}$ if $\beta > 0$, $1 + (\log 1/r)^+$, if $\beta = 0$ and 1, if $\beta < 0$
$G_\varepsilon\phi$	see (IV.3.4)
$G(f,t)$	$\int_0^t \sup_x P_s f(x) ds$
$G(X)$	$\cup_{\delta>0}\mathrm{cl}\{(t,x) : t \geq \delta,\ x \in S(X_t)\}$, the graph of X
$h-m$	the Hausdorff h-measure–see Section III.3
$h(r)$	Lévy's modulus function $(r\log(1/r))^{1/2}$
$h_d(r)$	$r^2 \log^+ \log^+ 1/r$ if $d \geq 3$, $r^2(\log^+ 1/r)(\log^+ \log^+ \log^+ 1/r)$ if $d = 2$
$H_t^{s,y}$	the H_t measure of $\{w : w = y$ on $[0,s]\}$, $s \leq t$, $y(\cdot) = y(\cdot \wedge s)$
$\overline{\mathcal{H}}^{bp}$	the bounded pointwise closure of $\mathcal{H}$
$\mathcal{H}_+$	the set of non-negative functions in $\mathcal{H}$
I	$\bigcup_{n=0}^{\infty} \mathbb{N}^{\{0,\dots,n\}} = \{(\alpha_0,\dots,\alpha_n) : \alpha_i \in \mathbb{N}, n \in \mathbb{Z}_+\}$
IBSMP	time inhomogeneous Borel strong Markov process–see after Lemma II.8.1
$I(f,t)$	stochastic integral of f on a Brownian tree–see Proposition V.3.2
$\mathcal{K}$	the compact subsets of $\mathbb{R}^d$
Lip_1	Lipschitz continuous functions with Lipschitz constant and supnorm ≤ 1
(LE)	Laplace functional equation–see prior to Theorem II.5.11
$(LMP)_\nu$	local martingale problem for Dawson-Watanabe superprocess with initial law ν–see prior to Theorem II.5.1
$L_t(X)$	the collision local time of $X = (X^1, X^2)$–see prior to Remarks IV.3.1
$\log^+(x)$	$(\log x) \vee e^e$
$L_W(\phi)$	the Laplace functional of the random measure W, i.e., $E(e^{-W(\phi)})$
$\mathcal{L}^2_{\mathrm{loc}}$	see after Lemma II.5.2
$M_1(E)$	space of probabilities on E with the topology of weak convergence
$M_F(E)$	the space of finite measures on E with the topology of weak convergence
$M_F^t(D)$	the set of finite measures on $D(E)$ supported by paths which are constant after time t
$\mathcal{M}_F$	the Borel σ-field on $M_F(E)$
$\mathcal{M}_{\mathrm{loc}}$	the space of continuous $(\mathcal{F}_t)$-local martingales starting at 0
(ME)	mild form of the nonlinear equation–see prior to Theorem II.5.11
$(MP)_{X_0}$	martingale problem for Dawson-Watanabe superprocess with initial state X_0 –see Proposition II.4.2
$\hat{\Omega}$	$\Omega \times C(\mathbb{R}^d)$
$\Omega_H[\tau,\infty)$	$\Big\{H_\cdot \in C\Big([\tau,\infty), M_F\big(D(E)\big)\Big) : H_t \in M_F^t(D) \quad \forall t \geq \tau\Big\}$
Ω_H	$\Omega_H[0,\infty)$
Ω_X	the space of continuous $M_F(E)$-valued paths
Ω_D	the space of cadlag $M_F(E)$-valued paths
$p_t(x)$	standard Brownian density
$p_t^x(y)$	$p_t(x-y)$
$\mathcal{P}$	the σ-field of $(\mathcal{F}_t)$-predictable subsets of $\mathbb{R}_+ \times \Omega$
$P_t^g\phi(x)$	$E^x\Big(\phi(Y_t)\exp\Big\{\int_0^t g(Y_s)\,ds\Big\}\Big)$
$\mathbb{P}_{X_0}$	the law of the DW superprocess on $(\Omega_X, \mathcal{F}_X)$ with initial state X_0 –see Theorem II.5.1
$\mathbb{P}_\nu$	the law of the DW superprocess with initial law ν

$\hat{\mathbb{P}}_T$	the normalized Campbell measure associated with K_T, i.e., $\hat{\mathbb{P}}_T(A\times B)=\mathbb{P}(1_A K_T(B))/m(1)$
(PC)	$x\mapsto P^x$ is continuous
(QLC)	quasi-left continuity, i.e., Y is a Hunt process–see Section II.2
$\mathbb{Q}_{\tau,m}$	the law of the historical process starting at time τ in state m–see Section II.8
$\mathcal{R}(I)$	$\bigcup_{t\in I} S(X_t)$, the range of X on I
$\overline{\mathcal{R}}(I)$	$\overline{\mathcal{R}(I)}$ is the closed range of X on I
$\mathcal{R}$	$\bigcup_{\delta>0} \overline{\mathcal{R}}\big([\delta,\infty)\big)$ is the range of X.
$S(\mu)$	the closed support of a measure μ
S_t	$S(X_t)$
$\mathcal{S}$	simple $\mathcal{P}\times\mathcal{E}$-measurable integrands–see after Lemma II.5.2
(SE)	strong form of nonlinear equation–see prior to Theorem II.5.11
$\underline{t}$	$[Nt]/N$
$\mathcal{T}_b$	bounded $(\mathcal{F}_t)_{t\ge\tau}$-stopping times
$\stackrel{ucb}{\to}$	convergence on E which is uniform on compacts and bounded on E
U_λ	the λ resolvent of a Markov process
$\stackrel{w}{\Rightarrow}$	weak convergence of finite (usually probability) measures
W_t	the coordinate maps on $D(\hat{E})$
$y/s/w$	the path equaling y up to s and $w(t-s)$ thereafter
$y^t(\cdot)$	$y(t\wedge\cdot)$
ζ_α	the lifetime of the α^{th} branch–see after Remark II.3.2