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The Spectral Theorem

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Foreword

This volume is based on lectures given at the Summer School for graduate students held at the Mathematical Institute, Nankai University, in Tianjin during June and July, 1985. It is a pleasure to thank my assistants, Mr. Gu Chi of Nanking University and Mr. Zheng Xue-an of Anhui University, who helped conduct the class and took notes; and to express my appreciation to Professor S. S. Chern for his encouragement in writing for this series.

The subject of the lectures was loosely defined, but could be described as the Spectral Theorem and questions related to it.

The first chapter contains the author's approach to the construction of the multiplicity function of a spectral measure. The main result asserts that every spectral measure is unitarily equivalent to one of a certain concrete type. The notion of range function gives us a canonical form for the spectral measure that is geometrically simple. We use this representation theorem in the rest of the book.

The multiplicity theorem is proved in [A], [Ha 2], [N], and [St]. The author hopes that this version is more intuitive than others. The proofs here are simplified by the fact that our spectral measure is defined on a countably generated field and acts in separable Hilbert space.

Chapter 2 is a proof of the spectral theorem itself, in the three usual versions. The theorem is often stated and proved first for (bounded) self-adjoint operators, and then Stone's theorem for continuous unitary groups is obtained by some device that may not seem very natural. We have chosen to prove Stone's theorem first and derive the versions for self-adjoint and normal operators from it. We use an idea from [RS]: the representation of the line R is integrated to give a representation of $L^1(R)$. By extending this homomorphism sufficiently the spectral projections can be found. (Riesz and Sz.-Nagy only apply the idea to the iterates of a single unitary operator.) Our proof can be extended to other locally compact abelian groups, and indeed the version for R^2 is used to prove the spectral theorem for normal operators.

In Chapter 3 two versions of Bochner's theorem are proved on the line. Bochner's original theorem, about continuous positive definite functions, is well known and accessible, but the second formulation, about functions positive definite in an integral sense, is not. It is ascribed in [E] to Gelfand and Raikov, but [R] is earlier and relevant. We prove the two theorems by means of the same ideas. Kolmogorov's extension theorem is derived as a corollary of Bochner's theorem for the infinite product of integer groups.

Chapter 4 begins with results from a recent paper of the author [He 3]. They rely on Bochner's theorem and thus on the spectral theorem; besides, they provide an introduction to cocycles, which are the subject of the rest of the book. We present an interesting application of ideas about additive cocycles to Diophantine approximation, following papers of H. Kesten and K. Petersen. We use the results on the distribution of values of cocycles to give new and quite simple proofs of theorems due to Hamachi, Oka and Osikawa, and to Moore and Schmidt.

Chapter 5 treats multiplicative cocycles. We are investigating unitary representations of a discrete subgroup Γ of the line in $L^2(R)$ that satisfy the Weyl commutation relation with the group of exponential multiplications. Under hypotheses of continuity we obtain theorems of Beurling and of Stone and von Neumann, both of which state, in disguised language, that certain cocycles are coboundaries. Beurling's theorem says that every simply invariant subspace of $L^2(R)$ has the form $q.H^2(R)$, where q is a unitary function on the line. The analogous result on the circle has an easy proof in Hilbert space, and the theorem on the line is ordinarily obtained by conformal mapping from the circle. However in the translation there are real function-theoretic difficulties. The proof here proceeds entirely on the line, where the crucial point is to show that a certain cocycle is a coboundary. This cocycle is almost trivially cohomologous to a periodic cocycle, which can be treated easily. Thus we seem to have a new proof in which the reduction to the periodic case is simple.

The theorem of Stone and von Neumann asserts the uniqueness of solutions of the Weyl commutation relation. This is an extension of Beurling's theorem in two ways. First, a spectral measure has to be shown to be absolutely continuous. Second, the cocycle that must be shown to be a coboundary has values that are unitary operators in a Hilbert space, instead of complex numbers. On the other hand, Beurling's theorem needs function-theoretic information (the fact that a function of analytic type cannot vanish on a set of positive measure unless it vanishes identically) that is not relevant to the theorem of Stone and von Neumann. Both these theorems are widely known, but the fact that they are closely related does not seem to be common knowledge. The ideas of these sections have their genesis in a seminal paper of G. W. Mackey [M].

Finally, a series of results of the author about the spectrum of unitary operators derived from multiplicative cocycles is presented. Hard problems remain to be studied. Good luck to my students from Tianjin!

HH

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