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Transformation Groups and Algebraic K-Theory

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0. Introduction.

The main goal of algebraic topology is the translation of problems and phenomena from geometry to algebra. In favourable cases we obtain a computable algebraic invariant which decides a given geometric question. A classical example is the classification of compact connected closed orientable surfaces by the genus.

This book is devoted to the connection between transformation groups and algebraic K-theory. We shall construct invariants such as the equivariant finiteness obstruction, Whitehead torsion, and Reidemeister torsion taking values in algebraic K-groups. We define injections or isomorphisms to the algebraic K-groups from groups such as finiteness obstruction groups, Whitehead groups, representation rings, homotopy representation groups or units of the Burnside ring. These are used to answer questions of the following type:

When is a finitely dominated G-CW-complex G-homotopy equivalent to a finite G-CW-complex? Under which conditions is a G-homotopy equivalence between finite G-CW-complexes simple? Is a given equivariant h-cobordism trivial? When are two semilinear G-discs G-diffeomorphic? Under which conditions are the unit spheres of two orthogonal G-representations G-diffeomorphic? When are two oriented G-homotopy representations oriented G-homotopy equivalent? Is a given oriented G-homotopy representation oriented G-homotopy equivalent to the unit sphere of a complex G-representation?

These questions will be treated in detail. They are related to the general problem of classifying group actions on manifolds. This problem and in particular its connections to algebraic K-theory are the basic motiviation for this book. We concentrate on developing the algebra. The algebraic tools and techniques presented here have applications to G-manifolds besides the one to the questions above . They will not be worked out, as this would exceed the scope of this book, but are discussed in the comments.

Roughly speaking, most of the material of chapter I can be found in the literature whereas chapters II and III mainly contain unpublished work. The study of modules over a category was initiated by Bredon [1967], where an equivariant obstruction theory for extending G-maps was established, and by tom Dieck [1981], where the equivariant finiteness obstruction and the diagonal product formula were studied for finite groups

and simply connected fixed point sets. The author wants to express his deep gratitude to Prof. Tammo tom Dieck for his encouragement and generous help.

The book is based on a course given by the author in the winter term 1986/87 and on the author's Habilitationsschrift, Göttingen 1989.

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We briefly summarize the main results and constructions.

0.1. Modules over a category.

Let Γ be a EI-category, i.e., Γ is a small category whose endomorphisms are isomorphisms. A $R\Gamma$-module M is a contravariant functor $\Gamma \longrightarrow$ R-MOD into the category of modules over the commutative ring R. The functor category MOD-$R\Gamma$ of $R\Gamma$-modules is abelian. We reduce the study of $R\Gamma$-modules, their K-theory and homological algebra to the study of $R[x]$-modules for $x \in \operatorname{Ob}\Gamma$ and $R[x]$ the group ring $R[\operatorname{Aut}(x)]$ by the Cofiltration Theorem 9.39. and Filtration Theorem 16.8. The Cofiltration resp. Filtration Theorem assigns to a projective $R\Gamma$-module P of finite tpye resp. $R\Gamma$-module M of finite length a natural cofiltration

$$P = P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_o = \{0\}$$

resp. natural filtration

$$\{0\} = M_o \longrightarrow M_1 \longrightarrow \dots \longrightarrow M_n = M$$

such that the kernel of $P_i \longrightarrow P_{i-1}$ resp. cokernel of $M_i \longrightarrow M_{i+1}$ can be expressed in terms of $R[x]$-modules S_xP resp. Res_xM which themselves are naturally constructed from P resp. M. The Cofiltration Theorem implies the Splitting Theorem 10.34. for algebraic K-theory of $R\Gamma$-modules

$$K_n(R\Gamma) = \bigoplus_{\bar{x} \in \operatorname{Is}\Gamma} K_n(R[x])$$

where $\bar{x}$ runs over the set $\operatorname{Is}\Gamma$ of isomorphism classes of objects and $n \in \mathbf{Z}$. As a special case we obtain the well-known splitting of the equivariant Whitehead group of a G-space. For finite Γ and R a field of characteristic 0 the Filtra-

tration Theorem gives a second Splitting Theorem 16.29. for algebraic K-theory of $R\Gamma$-modules. These two Splitting Theorems are related by a K-theoretic Moebius inversion 16.29. In geometry this corresponds to switching between the isovariant and equivariant setting, or between the two stratifications $\{X_H \mid H \subset G\}$ and $\{X^H \mid H \subset G\}$ of a G-space X. Besides the K-theoretic application we also obtain a computation of Ext-groups $\mathrm{EXT}^n_{R\Gamma}(M,N)$ by a spectral sequence whose E_2-term is given by Ext-groups over the various group rings $R[x]$ (see 17.18. and 17.28.). We introduce and study generalized Swan homomorphisms in section 19.

The algebra of $R\Gamma$-modules for Γ the discrete fundamental category $\Pi/(G,X)$ (see 8.15.) of a G-space X is the main ingredient for constructing and computing certain algebraic invariants of G-spaces and the K-groups in which they take values.

0.2. Invariants for G-spaces.

Here is a list of the most important invariants we will construct for G-spaces and G-maps.

name	symbol	value group	defined for	page
Euler characteristic	$\chi^G(X)$	$U^G(X)$ resp. $U(G)$	finitely dominated G-space X	100, 278, 360
multiplicative Euler characteristic	$h\chi(X)$	$\prod_{(H)} \mathbb{Q}^*/\mathbf{Z}^*$	finitely dominated G-space X	368
	$m\chi(X)$	$\prod_{(H)} \mathbb{Q}^*/\mathbf{Z}^*$	special G-space X	368
	$h\chi(X)_{1/m}$	$\bar{C}(G)^* = \prod_{(H)} \mathbf{Z}/\|G\|^*$	finitely dominated G-space X	387
	$h\chi(f)_{1/m}$	$\bar{C}(G)^*$	G-map between finitely dominated G-spaces	387
finiteness obstruction	$o^G(X)$	$K_o(\mathbf{Z}\Pi/(G,X))$ resp. $K_o(\mathbf{Z}\mathrm{Or}G)$	finitely dominated G-space X	278, 360
reduced finiteness obstruction	$\tilde{o}^G(X)$	$\tilde{K}_o(\mathbf{Z}\Pi/(G,X))$ resp. $\tilde{K}_o(\mathbf{Z}\mathrm{Or}G)$	finitely dominated G-space X	278, 360
	$w^G(X)$	$Wa^G(X)$		52

name	symbol	value group	defined for	page
(equivariant) Whitehead torsion	$\tau^G(f)$	$Wh(\mathbf{Z}\Pi/(G,Y))$ resp. $Wh(\mathbf{Z}Or\,G)$	G-homotopy equivalence of finite G-CW-complexes resp. G-manifolds $f : X \longrightarrow Y$	284, 360
	$\tau^G_{geo}(f)$	$Wh^G_{geo}(Y)$		68
isovariant Whitehead torsion	$\tau^G_{Iso}(B,M,N)$	$Wh^G_{Iso}(M)$	Isovariant h-cobordism (B,M,N)	85
Reidemeister torsion	$\rho^G(X)$	$Wh(\mathbb{Q}Or\,G)$	Finite G-CW-complex with round structure X	362
	$\rho^G(M)$	$Wh(\mathbb{R}Or\,G)$	M a closed Riemannian G-manifold satisfying 18.43.	375
	$\rho^G_{PL}(M)$	$K_1(\mathbb{R}G)^{\mathbf{Z}/2}$	M a Riemannian G-manifold	376
reduced Reidemeister torsion	$\bar{\rho}^G(X)$	$\dfrac{K_1(\mathbb{Q}Or\,G)}{K_1(\mathbf{Z}_{(\|G\|)}Or\,G)}$	Finitely dominated G-space X with round structure	363
Poincaré torsion	$\rho^G_{PD}(M)$	$K_1(\mathbb{R}G)^{\mathbf{Z}/2}$	Riemannian G-manifold M	377

We compute the value groups in terms of algebraic K-groups of certain group rings and state sum, product, diagonal product, join and restriction formulas. The reduced finiteness obstruction is the obstruction for a finitely dominated G-space X to be G-homotopy equivalent to a finite G-CW-complex. The Whitehead torsion is the obstruction of a G-homotopy equivalence of finite G-CW-complexes to be simple. Both invariants are defined geometrically and algebraically and these two approaches are identified by isomorphisms $Wa^G(X) \longrightarrow \tilde{K}_0(\mathbf{Z}\Pi/(G,X))$ and $Wh^G_{geo}(X) \longrightarrow Wh(\mathbf{Z}\Pi/(G,X))$. Certain relations between these invariants are established. Roughly speaking, Whitehead torsion is the difference of Reidemeister torsion, the reduced Reidemeister torsion is a refinement of the finiteness obstruction.

0.3. Maps between geometric groups and K-groups.

We give a list of maps relating geometrically defined groups to algebraic K-groups. They connnect geometry with algebraic K-theory. We denote injections by $\rightarrowtail$ and isomorphisms by $\xrightarrow{\cong}$:

0.4. Applications to geometry

We restate the Isovariant s-Cobordism Theorem 4.42. saying that isovariant h-cobordisms are classified by their isovariant Whitehead torsion. We relate the isovariant and equivariant setting by an homomorphism $\Phi : Wh^G_{Iso}(M) \longrightarrow Wh^G(M)$. Provided that the weak gap conditions 4.49. are satisfied, we show that Φ is injective and determine its image and thus get the Equivariant s-Cobordism Theorem 4.51. We give counter-examples to the Equivariant s-Cobordism Theorem 4.51. without the weak gap hypothesis in Example 4.56.

We prove for a finite group G of odd order that the transfer on K_o and Wh induced by the sphere bundle of a G-vector bundle vanishes under mild conditions (see 15.29.). These transfer maps appear e.g. in the comparison of isovariant and equivariant Whitehead groups and in the involution defined on them by reversing h-cobordisms.

We construct an homomorphism $\rho^G_{\mathbf{R}} : Rep_{\mathbf{R}}(G) \longrightarrow Wh(\mathbb{Q}Or\,G)$, $[V] \longrightarrow \rho^G(S(V \oplus V))$ and prove injectivity in 18.38. Hence spheres of real G-representations are classified up to G-diffeomorphism by Reidemeister torsion. This reproves de Rham's theorem that two

G-representations are RG-isomorphic if and only if their spheres are G-diffeomeorphic.

A G-homotopy representation X is a finite-dimensional G-CW-complex such that $X^H = S^{n(H)}$ holds for $H \subset G$. Given two G-homotopy representations X and Y with $\dim X^H = \dim Y^H$ for all $H \subset G$, we want to determine the set $[X,Y]^G$ of G-homotopy classes of G-maps between them. If we have choosen a coherent orientation, then

$$\mathrm{DEG} : [X,Y]^G \longrightarrow \prod_{(H)} \mathbf{Z} , \quad [f] \longrightarrow (\deg f^H)_{(H)}$$

is an injection. We give in Theorem 20.38. a set of congruences describing the image of DEG and hence $[X,Y]^G$ which can be computed from the difference of the reduced Reidemeister torsion $\bar{\rho}^G(Y) - \bar{\rho}^G(X)$ by generalized Swan homomorphisms. In particular we get that G-homotopy representations are classified up to oriented G-homotopy equivalence by an absolute invariant, the reduced Reidemeister torsion.

0.5. On the concept of the book.

We have tried to keep the book fairly self-contained. We give the definitions, results and proofs in full generality and illustrate them by examples. At the end of each section there is a comment where the material of the section is put into context with the work of other mathematicians, further applications are discussed and additional references are given. More information and results are contained in the exercises. We advise the reader to at least read through them.

This expansive way of writing means that the sections contain much more material and results in much larger generality than needed for the following sections of specific applications. Therefore we have tried to give the reader, who is only interested in a specific question, the possibility to pick out a single section and read it without knowing the others. Here is some advice for such a reader.

The chapters II and III are independent of chapter I. If one is interested in the algebra only, one may skip chapter I completely.

In chapter I one may begin with one of the sections 3, 4, or 5 directly as they are independent of one another and sections 1 and 2 are quite elementary.

Nearly all notions and results are stated for Lie groups G and proper G-actions without any assumptions about the connectivity of the fixed point sets. The notational and technical difficulties decrease considerably if G is a finite group and the fixed point sets are empty or simply connected. In this case a summary of the invariants defined for G-spaces in chapter II is given in section 18 including their basic properties. Moreover, section 8 is in this case of no importance, as everything takes place over the orbit category. In particular this restriction does no harm if one studies G-homotopy representations.

If one is interested in the finiteness obstruction and torsion only over the group ring resp. for the universal covering of a G-space without group action, one may directly begin with section 11 and 12 thinking of $R\Gamma$ as RG, and similarly for the material about the Swan homomorphism for group rings and its lifting in section 19.

An experienced reader can start with section 18 without having looked at the previous sections since the necessary input from them is reviewed in the beginning of section 18. Although section 20. makes use of section 18 and 19, section 20 can be read without knowing section 18 and 19 because only the formal properties of Reidemeister torsion and Swan homomorphism but not their explicit constructions are needed.

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