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Lectures on Symplectic Geometry

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Foreword

These notes approximately transcribe a 15-week course on symplectic geometry I taught at UC Berkeley in the Fall of 1997.

The course at Berkeley was greatly inspired in content and style by Victor Guillemin, whose masterly teaching of beautiful courses on topics related to symplectic geometry at MIT, I was lucky enough to experience as a graduate student. I am very thankful to him!

That course also borrowed from the 1997 Park City summer courses on symplectic geometry and topology, and from many talks and discussions of the symplectic geometry group at MIT. Among the regular participants in the MIT informal symplectic seminar 93–96, I would like to acknowledge the contributions of Allen Knutson, Chris Woodward, David Metzler, Eckhard Meinrenken, Elisa Prato, Eugene Lerman, Jonathan Weitsman, Lisa Jeffrey, Reyer Sjamaar, Shaun Martin, Stephanie Singer, Sue Tolman and, last but not least, Yael Karshon.

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Berkeley, November 1998
and Lisbon, September 2000

Ana Cannas da Silva

Contents

Part XI Symplectic Toric Manifolds

Introduction

The goal of these notes is to provide a fast introduction to symplectic geometry.

A symplectic form is a closed nondegenerate 2-form. A symplectic manifold is a manifold equipped with a symplectic form. Symplectic geometry is the geometry of symplectic manifolds. Symplectic manifolds are necessarily even-dimensional and orientable, since nondegeneracy says that the top exterior power of a symplectic form is a volume form. The closedness condition is a natural differential equation, which forces all symplectic manifolds to being locally indistinguishable. (These assertions will be explained in Lecture 1 and Homework 2.)

The list of questions on symplectic forms begins with those of existence and uniqueness on a given manifold. For specific symplectic manifolds, one would like to understand the geometry and the topology of special submanifolds, the dynamics of certain vector fields or systems of differential equations, the symmetries and extra structure, etc.

Two centuries ago, symplectic geometry provided a language for classical mechanics. Through its recent huge development, it conquered an independent and rich territory, as a central branch of differential geometry and topology. To mention just a few key landmarks, one may say that symplectic geometry began to take its modern shape with the formulation of the Arnold conjectures in the 60's and with the foundational work of Weinstein in the 70's. A paper of Gromov [49] in the 80's gave the subject a whole new set of tools: pseudo-holomorphic curves. Gromov also first showed that important results from complex Kähler geometry remain true in the more general symplectic category, and this direction was continued rather dramatically in the 90's in the work of Donaldson on the topology of symplectic manifolds and their symplectic submanifolds, and in the work of Taubes in the context of the Seiberg-Witten invariants. Symplectic geometry is significantly stimulated by important interactions with global analysis, mathematical physics, low-dimensional topology, dynamical systems, algebraic geometry, integrable systems, microlocal analysis, partial differential equations, representation theory, quantization, equivariant cohomology, geometric combinatorics, etc.

As a curiosity, note that two centuries ago the name *symplectic geometry* did not exist. If you consult a major English dictionary, you are likely to find that *symplectic*

is the name for a bone in a fish's head. However, as clarified in [105], the word *symplectic* in mathematics was coined by Weyl [110, p.165] who substituted the Latin root in *complex* by the corresponding Greek root, in order to label the symplectic group. Weyl thus avoided that this group connote the complex numbers, and also spared us from much confusion that would have arisen, had the name remained the former one in honor of Abel: *abelian linear group*.

This text is essentially the set of notes of a 15-week course on symplectic geometry with 2 hour-and-a-half lectures per week. The course targeted second-year graduate students in mathematics, though the audience was more diverse, including advanced undergraduates, post-docs and graduate students from other departments. The present text should hence still be appropriate for a second-year graduate course or for an independent study project.

There are scattered short exercises throughout the text. At the end of most lectures, some longer guided problems, called homework, were designed to complement the exposition or extend the reader's understanding.

Geometry of manifolds was the basic prerequisite for the original course, so the same holds now for the notes. In particular, some familiarity with de Rham theory and classical Lie groups is expected.

As for conventions: unless otherwise indicated, all vector spaces are real and finite-dimensional, all maps are smooth (i.e., C^∞) and all manifolds are smooth, Hausdorff and second countable.

Here is a brief summary of the contents of this book. Parts I–III explain classical topics, including cotangent bundles, symplectomorphisms, lagrangian submanifolds and local forms. Parts IV–VI concentrate on important related areas, such as contact geometry and Kähler geometry. Classical hamiltonian theory enters in Parts VII–VIII, starting the second half of this book, which is devoted to a selection of themes from hamiltonian dynamical systems and symmetry. Parts IX–XI discuss the moment map whose preponderance has been growing steadily for the past twenty years.

There are by now excellent references on symplectic geometry, a subset of which is in the bibliography. However, the most efficient introduction to a subject is often a short elementary treatment, and these notes attempt to serve that purpose. The author hopes that these notes provide a taste of areas of current research, and will prepare the reader to explore recent papers and extensive books in symplectic geometry, where the pace is much faster.