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# Diophantine Approximation

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# Preface

Diophantine Approximation is a branch of Number Theory having its origins in the problem of producing "best" rational approximations to given real numbers. Since the early work of Lagrange on Pell's equation and the pioneering work of Thue on the rational approximations to algebraic numbers of degree $\geq 3$, it has been clear how, in addition to its own specific importance and interest, the theory can have fundamental applications to classical diophantine problems in Number Theory. During the whole 20th century, until very recent times, this fruitful interplay went much further, also involving Transcendental Number Theory and leading to the solution of several central conjectures on diophantine equations and class number, and to other important achievements. These developments naturally raised further intensive research, so at the moment the subject is a most lively one.

This motivated our proposal for a C.I.M.E. session, with the aim to make it available to a public wider than specialists an overview of the subject, with special emphasis on modern advances and techniques. Our project was kindly supported by the C.I.M.E. Committee and met with the interest of a large number of applicants; forty-two participants from several countries, both graduate students and senior mathematicians, intensively followed courses and seminars in a friendly and co-operative atmosphere.

The main part of the session was arranged in four six-hours courses by Professors D. Masser (Basel), H. P. Schlickewei (Marburg), W. M. Schmidt (Boulder) and M. Waldschmidt (Paris VI).

This volume contains expanded notes by the authors of the four courses, together with a paper by Professor Yu. V. Nesterenko (Moscow) – who was unable to accept our invitation to give an expected fifth course – concerning recent work by Matveev.

We shall now briefly illustrate the corresponding contents.

Masser's contribution concerns, roughly speaking, the modern theory of heights, starting with the most basic notions and then turning to the more sophisticated context of algebraic groups. This ample overview describes fundamental results and techniques in the subject, together with applications to transcendence problems. Masser also outlines the transcendence theory of elliptic logarithms and abelian functions (which he originally developed), and its important recent consequences toward outstanding diophantine problems on curves and abelian varieties.

Nesterenko's article is devoted to the proof of the nowadays best known lower bounds in Baker's theory of linear forms in logarithms of algebraic numbers. With the aim of stressing the new ideas introduced by Matveev, the

author concentrates on a situation slightly simpler in detail than the most general one, but containing all the important features of the methods.

Schlickewei deals with the celebrated Subspace Theorem. This result, originally discovered by W. M. Schmidt, is a far-reaching extension of Roth's Theorem on the approximations of an algebraic number by rationals, also covered in the lectures. Schlickewei describes the most recent sharpenings (such as the "absolute version"), obtained mainly in joint work by himself and J.-H. Evertse. Finally, he presents here his very recent work on a version of the theorem for approximation by algebraic numbers of bounded degree (obtained jointly with H. Locher).

Schmidt's article concerns the diophantine theory of linear recurrences, whose famous prototype is the Fibonacci sequence. He gives a general survey of the most important problems, methods and results, involving also $S$-unit equations and intersections of varieties with finitely generated multiplicative groups. In particular, he also illustrates the general strategy underlying his recent solution of an outstanding conjecture in the field; namely, *the zero-multiplicity of a non-degenerate linear recurrence is bounded only in terms of the "length" of the recurrence.*

Waldschmidt's contribution is on transcendence and linear independence over $\overline{\mathbb{Q}}$ of logarithms of algebraic numbers. Starting with Lindemann's classical theorems on the exponential function, he proceeds with the sophisticated results by A. Baker, which yield fundamental applications to effective diophantine analysis. Waldschmidt describes several approaches to the technically complicated proofs, clarifying the main ideas underlying methods which may confound the non-expert. He also details certain modern devices to obtain the best numerical bounds for the involved quantities.

The topics presented in such fine lecture notes incorporate many of the most fundamental methods and applications of Diophantine Approximation, giving an extremely broad viewpoint, precious for both beginners and experts. Also, the style of exposition has little in common with other contributions to the topic and the volume substantially enriches the existing literature.

It is a pleasure for us to thank the authors for their difficult work in coordinating the respective contributions, for their efforts in explaining the subtle points in the simplest and most effective style, and for working out these beautiful papers. We also thank the participants, whose enthusiasm was fundamental for the success of the session.

Finally, the editors express their thanks to Carlo Viola for his valuable advice and help concerning both the organization of the session and the preparation of the present volume.

Francesco Amoroso Umberto Zannier

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# Contents