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L. Molnár

Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces

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To my family

Preface

Originally this work served as the author's dissertation for the scientific title "Doctor of the Hungarian Academy of Sciences". To publish it in the form of a book has been initiated by the referees who formulated this suggestion in their reports.

In what follows we present a cross-section of recent research concerning preserver problems (both linear and non-linear) and local transformations (namely, local automorphisms and local isometries) defined on algebraic structures of linear operators and on function spaces.

Generally speaking, preserver problems concern the question of determining or describing the general form of all transformations of a given structure $\mathcal{X}$ which preserve

- a quantity attached to the elements of $\mathcal{X}$, or
- a distinguished set of elements of $\mathcal{X}$, or
- a given relation among the elements of $\mathcal{X}$,

etc. Such problems arise in most parts of mathematics. In fact, it turns out that in many cases the corresponding results provide important information on the automorphisms of the underlying structures. However, preserver problems are systematically studied only within the scope of matrix theory and, recently, within that of operator theory.

In this work we give a picture about some parts of that area of research by presenting several of our related results and referring to corresponding results of other authors.

The first group of questions we study here consists of preserver problems in the 'classical' sense, i.e., of problems which can be classified into that systematic study mentioned above. The second group is about preserver problems on certain fundamental structures of linear operators which appear in the mathematical foundations of quantum mechanics. Those results can also be viewed as the descriptions of certain automorphisms of the underlying quantum structures or, in other words, quantum mechanical symmetries.

The vague formulation of the basic problem concerning local transformations reads as follows. We are given a structure and a distinguished collection of transformations on it. Is it true that any transformation which locally belongs to that collection (i.e., which has the property that at every point its value coincides with the value of a transformation from our collection at that point) belongs to it also globally? If the answer to this question turns to be affirmative, then one can say that the collection under consideration is completely determined by its local actions. In this work we present some corresponding results which concern the collections (in fact, groups) of automorphisms and surjective isometries of operator algebras and function algebras. Although the ground problem of local transformations seems to have nothing to do with preserver problems, this is still an area of applications of the theory of preservers at least in part. In fact, as one can see in the proofs of the presented results we often use 'preserver arguments' and several particular preserver results.

Some words about the structure of the book which is very fixed and, hopefully, rather clear and logical.

The chapter 'Introduction' describes the considered research areas and presents the basic problems and important results obtained by other researchers. This is done in Sections 0.1, 0.3 and 0.5. These sections are followed by short surveys of our corresponding results which are included in the book. This is the content of Sections 0.2, 0.4 and 0.6. Section 0.7 collects the most frequently used notation and definitions.

Our results are presented in detail in the sections of Chapters 1, 2 and 3. Basically, they are taken from our corresponding papers but with a large number of modifications and revision, of course. These sections begin with short summaries describing the main results therein. After that more detailed introductions to the treated problems are presented and the precise formulations of the results are given. These subsections are followed by the complete proofs. At the end of each section some remarks are made on further results of ours or other authors, or on possible further directions of research.

In Appendix some basic results are collected that we use several times in Chapters 1, 2 and 3.

This was about the content of the book. Let me close the preface with some acknowledgements. In fact, it would have been much much harder to write this work without the support I have been lucky to get from the following three sources: my family, the Alexander von Humboldt Foundation and my friend Werner Timmermann. I am very grateful to all of them. To my family for the continuous love and for the support in the critical periods of my life. To the Alexander von Humboldt Foundation whose research fellowship I held during my stay at the Technical University of Dresden in the academic year 2002-2003 for the great possibility that for one complete year I could concentrate all of my energy on research and on the composition of this material. And to Werner who was my academic host in Dresden for the exceptionally warm hospitality including many illuminating discussions not only about mathematics but also about the matters of life.

Finally, special thanks are due to László Kérchy, Béla Nagy, Zoltán Sebestyén (referees of the original dissertation) and Werner Timmermann for the encouragement to publish this book, to the anonymous referees at Springer for their helpful comments and suggestions, and to Marina Reizakis from Springer for her kind assistance during the process of publication.

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Lajos Molnár

Contents

Introduction

0.1 Linear Preserver Problems

Particular preserver problems appear in many parts of mathematics. This is not surprising since it is a natural question in many contexts to ask that what are the transformations on a given structure which preserve 'something' (the meaning of this 'something' is of course well-defined in the problem under consideration and is in connection with the underlying structure). However, it seems that the only field in mathematics where preserver problems are studied systematically are matrix theory and, recently, its infinite dimensional variant, i.e., operator theory.

In fact, the so-called linear preserver problems (abbreviated as LPPs) represent one of the most active research areas in matrix theory. For surveys of the topic we refer to the papers [208, 136, 137]. According to the linear character of matrix theory, preserver problems mean here the characterizations of all linear transformations on a given linear space of matrices that leave certain functions, subsets, relations, etc. invariant. In the last decades there have been remarkable interest in similar problems also in the infinite dimensional case, i.e., when the underlying space consists of bounded linear operators (acting on a Hilbert space or on a Banach space) rather than matrices. For a corresponding survey see [29]. As in our work we are interested in problems of this latter kind, we are going to present the classification of linear preserver problems due to Li and Tsing [136] in such a setting.

Let $\mathcal{M}$ be a linear space of bounded linear operators on a Hilbert space. The first group of LPPs is concerned with the study of those linear transformations on $\mathcal{M}$ which preserve certain functions.

PROBLEM I. Let F be a (scalar-valued, vector-valued, or set-valued) given function on $\mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$F(\phi(A)) = F(A) \qquad (A \in \mathcal{M}). \tag{0.1.1}$$

To give an example for such a problem, we recall a well-known result of Frobenius from 1897 describing the general form of all determinant preserving linear maps on matrix algebras which is commonly considered as the first result on LPPs. Namely, Frobenius proved in [76] that if $\mathcal{M} = M_n(\mathbb{C})$ (the algebra of all $n \times n$ complex matrices) and $F(A) = \det A$, then every linear transformation ϕ which satisfies (0.1.1) is either of the form

$$\phi(A) = MAN \qquad (A \in \mathcal{M})$$

or of the form

$$\phi(A) = MA^{tr}N \qquad (A \in \mathcal{M})$$

for some nonsingular matrices $M, N \in M_n(\mathbb{C})$ with $\det MN = 1$. (A^{tr} stands for the transpose of A.)

Next, observe that the problem of describing the surjective linear isometries of operator algebras can also be classified into this group of problems. In fact, we define F to be the norm function. (In relation with this, we remark that one would be right when saying that the characterization of all isometries of a given metric space is also a preserver problem in a more general but similar sense. See the discussion after the classification of LPPs.) In his famous paper [117], Kadison described the surjective linear isometries of C^*-algebras. He proved that every such transformation can be written as a so-called Jordan *-isomorphism multiplied by a fixed unitary element. As we see from Theorem A.9 in Appendix, this general result implies that the surjective linear isometries of the Banach space $B(H)$ of all bounded linear operators acting on the Hilbert space H can be described in the following way. The surjective linear map $\phi : B(H) \to B(H)$ is an isometry if and only if there are unitary operators U, V on H such that ϕ is either of the form

$$\phi(A) = UAV \qquad (A \in B(H))$$

or of the form

$$\phi(A) = UA^{tr}V \qquad (A \in B(H))$$

(here and in what follows ^{tr} denotes the transpose of operators with respect to an arbitrary but fixed complete orthonormal system).

As another important preserver problem of the same type, we mention the problem of spectrum preserving transformations. Clearly, here $F(A)$ equals $\sigma(A)$, the spectrum of the operator A. In [107], Jafarian and Sourour proved that if $\phi : B(H) \to B(H)$ is a surjective linear transformation with the property that $\sigma(\phi(A)) = \sigma(A)$ holds for every $A \in B(H)$, then there is an invertible operator $T \in B(H)$ such that ϕ is either of the form

$$\phi(A) = TAT^{-1} \qquad (A \in B(H))$$

or of the form

$$\phi(A) = TA^{tr}T^{-1} \qquad (A \in B(H)).$$

(In fact, in [107] the authors presented a similar result concerning Banach space operators not only Hilbert space operators.) As for the much more general problem of spectrum preserving surjective linear maps between semi-simple Banach algebras we refer to [9] and some of the references therein as well.

Now we turn to the second group of LPPs which concern linear transformations that preserve certain subsets.

Problem II. Let $\mathcal{S}$ be a given subset of $\mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$A \in \mathcal{S} \Longrightarrow \phi(A) \in \mathcal{S} \qquad (A \in \mathcal{M})$$

or satisfy

$$A \in \mathcal{S} \Longleftrightarrow \phi(A) \in \mathcal{S} \qquad (A \in \mathcal{M}).$$

In the first case we say that ϕ preserves the elements of $\mathcal{S}$ in one direction (or, more simply, that ϕ preserves the elements of $\mathcal{S}$) while in the second case we say that ϕ preserves the elements of $\mathcal{S}$ in both directions.

When giving examples for this type of LPPs, one should not forget to mention the famous Kaplansky's problem on invertibility preservers although it concerns general Banach algebras not merely operator algebras. Motivated by the results in [65, 151] on invertibility preserving linear maps of matrix algebras and by the famous Gleason-Kahane-Żelazko theorem on the characterization of multiplicative linear functionals of commutative Banach algebras, Kaplansky [122] asked when must invertibility preserving linear transformations on algebras be Jordan homomorphisms (for the definition see the section Notation). For general Banach algebras the answer turned to be negative. In fact, borrowing a simple example from [234], consider the algebra of 3×3 upper triangular complex matrices and the transformation

$$\phi : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \longmapsto \begin{bmatrix} a_{11} & a_{13} & a_{12} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}.$$

Obviously, this is a surjective unital linear map from a Banach algebra onto itself which preserves invertibility in both directions but it is not a Jordan homomorphism since for the matrix

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

we have $\phi(C^2) = 0$, $\phi(C)^2 \neq 0$. Therefore, the question was modified and Kaplansky's Problem now reads as follows.

Kaplansky's Problem. Let $\mathcal{A}, \mathcal{B}$ be semi-simple Banach algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective linear transformation with the properties that $\phi(1) = 1$ (i.e., ϕ is unital) and $\phi(A) \in \mathcal{B}$ is invertible whenever so is $A \in \mathcal{A}$ (i.e., ϕ preserves invertibility in one direction). Is it true that ϕ is necessarily a Jordan homomorphism?

Observe that this problem is in an intimate relation with spectrum preservers since if a linear transformation $\phi : \mathcal{A} \to \mathcal{B}$ is unital, then it preserves invertibility in both directions if and only if $\sigma(\phi(A)) = \sigma(A)$ $(A \in \mathcal{A})$ while it preserves invertibility in one direction if and only if $\sigma(\phi(A)) \subset \sigma(A)$ $(A \in \mathcal{A})$. Although a lot of work has been devoted to the solution of Kaplansky's Problem and several important results have been obtained, the problem is still open in its full generality. We mention only two breakthrough steps on the way to a possible solution. The first one is due to Sourour who proved in [234] that the conclusion in Kaplansky's Problem is true if $\mathcal{A}, \mathcal{B}$ are full operator algebras on Banach spaces. The second one is due to Aupetit who showed in [10] that every surjective unital linear map between von Neumann algebras which preserves invertibility in both directions is a Jordan isomorphism.[1]

Another important problem from the same group of LPPs is the one of unitary preservers. In [149], Marcus described the form of all linear transformations on $M_n(\mathbb{C})$ which preserve the unitary matrices in one direction. In [219], Russo and Dye generalized this result for arbitrary C^*-algebras showing that every linear transformation $\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras which maps the unitary elements into unitary elements can be written as a unital Jordan *-homomorphism multiplied by a fixed unitary element. (We also refer to the paper [213] of Rais where the particular case $\mathcal{A} = \mathcal{B} = B(H)$ was treated.)

Our last example here is about linear transformations preserving operators with fixed rank. LPPs concerning rank are among the most important preserver problems. The reason is that it turns out in many cases that, with some effort, the original LPP under consideration can be reduced to such a problem. Among a number of relevant examples we only refer to the arguments in [107, 234] where such reductions were the clues of the proofs. The probably most fundamental result concerning rank preservers on operator algebras is due to Hou. In the paper [104] he described the general form of all weakly continuous linear maps on the whole operator algebra of a Banach space which preserve the rank-one operators. (We also refer to [203] for some results on additive maps preserving rank-one operators.)

The third group of LPPs concerns linear transformations preserving certain relations.

[1] Added in revision: We remark that one of the referees of the book pointed out that the same conclusion holds also for bijective unital maps preserving invertibility only in one direction. This follows from the discussion in [10, Remark 2.7]. Next, we remark that in the paper [60] Cui and Hou could strengthen the results of Aupetit. Namely, they proved that every surjective unital invertibility preserving linear map from a von Neumann algebra onto a semi-simple Banach algebra is a Jordan homomorphism.

PROBLEM III. Let $\sim$ be a relation on $\mathcal{M}$. Characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$A \sim B \Longrightarrow \phi(A) \sim \phi(B) \qquad (A, B \in \mathcal{M})$$

or satisfy

$$A \sim B \Longleftrightarrow \phi(A) \sim \phi(B) \qquad (A, B \in \mathcal{M})$$

In the first case we say that ϕ preserves the relation $\sim$ in one direction (or, more simply, that ϕ preserves the relation $\sim$) while in the second case we say that ϕ preserves $\sim$ in both directions.

A problem which should be certainly mentioned here is the problem of preserving commutativity. Although there are many results on the problem, this is still an active research topic. (For a recent remarkable achievement of the recent years see [202].) The basic result concerning commutativity preserving linear transformations on operator algebras is due to Omladič. In the paper [201] he described the structure of all bijective linear transformations on the whole operator algebra of a Banach space which preserve the commutativity in both directions. In the Hilbert space setting his result reads as follows. Let H be a Hilbert space of dimension at least 3 and let $\phi : B(H) \to B(H)$ be a bijective linear transformation which preserves commutativity in both directions. Then there exist a nonzero scalar λ, an invertible operator $T \in B(H)$ and a linear functional f on $B(H)$ such that ϕ is either of the form

$$\phi(A) = \lambda TAT^{-1} + f(A)I \qquad (A \in B(H))$$

or of the form

$$\phi(A) = \lambda TA^{tr}T^{-1} + f(A)I \qquad (A \in B(H)).$$

(We remark that the key step in Omladič's proof is to show that the transformation under consideration has a certain rank preserving property; see the last paragraph in the discussion of Problem II.)

To give another example, we mention the problem of zero product preserving mappings. These are the linear transformations $\phi : \mathcal{M} \to \mathcal{M}$ which satisfy

$$AB = 0 \Longleftrightarrow \phi(A)\phi(B) = 0 \qquad (A, B \in \mathcal{M}).$$

(We remark that similar transformations on function algebras are usually considered under the name 'disjointness preserving maps' or 'separating maps'.) For recent results on such transformations defined on operator algebras, we refer to [6], [58] and [105] (also see [50] for some general algebraic results). One of the statements in [105] asserts that if a surjective linear map $\phi : B(H) \to B(H)$ preserves zero product in both directions, then it is of the form

$$\phi(A) = \lambda TAT^{-1} \qquad (A \in B(H))$$

where λ is a nonzero scalar and $T \in B(H)$ is an invertible operator. For an interesting non-linear extension of this result we refer to last section of the paper [229].

Finally, the fourth group of LPPs concerns linear transformations which commute with certain maps on $\mathcal{M}$.

Problem IV. Given a map $F : \mathcal{M} \to \mathcal{M}$, characterize those linear transformations ϕ on $\mathcal{M}$ which satisfy

$$F(\phi(A)) = \phi(F(A)) \qquad (A \in \mathcal{M}).$$

Although in this case we sometimes say that ϕ preserves F, this is not to be confused with Problem I.

To mention an interesting problem of this kind, we refer to the paper [48], where all the linear transformations on a full matrix algebra were described which preserve the kth power (here $F(A) = A^k$, k being a fixed integer not less than 2). The importance of that problem lies in the fact that when $k = 2$, the corresponding preservers are exactly the Jordan homomorphisms of the underlying algebra. Using deep algebraic techniques, an analogous problem was solved in [24] concerning additive maps of general prime rings. As a very particular case of the result presented there, we obtain that if $\phi : B(H) \to B(H)$ is a surjective linear transformation with the property $\phi(A^k) = \phi(A)^k$ ($A \in B(H)$), then there exist an $(k-1)$th root of unity λ and an invertible operator $T \in B(H)$ such that ϕ is either of the form

$$\phi(A) = \lambda TAT^{-1} \qquad (A \in B(H))$$

or of the form

$$\phi(A) = \lambda TA^{tr}T^{-1} \qquad (A \in B(H)).$$

The problem of linear transformations on operator algebras preserving the absolute value (here $F(A) = |A|$) also belongs to this group of LPPs. For corresponding results we refer to [159, 211, 210] where the problem concerning merely additive (not necessarily linear) transformations were considered and we also mention the paper [18].

This is the classification scheme of linear preserver problems á la Li and Tsing [136]. It should be clear that in the formulations of the problems above it is not really essential to assume that the underlying structures are operator algebras or that the transformations what we consider are all linear. We mean that analogous problems can be raised on arbitrary structures concerning transformations of any kind. In fact, in that way the territory of preserver problems can be enormously enlarged and many problems from very different parts of mathematics can be regarded as preserver problems.

For a very broad class of transformations that appear in every part of mathematics and can be viewed as preservers in some sense, we refer to morphisms as transformations on a given structure which preserve ‘something’ that is peculiar to their domain. Let us mention some specific examples, for instance, from geometry.

First we recall Felix Klein and his Erlanger Programm. In his famous talk in 1872 on the occasion that he was appointed to a professorship in Erlangen, Klein presented a new unified approach to geometry. In his view, geometry is the study of the properties of a space that are invariant under a particular group of transformations. In that way, according to the inclusions among those transformation groups, he could make a hierarchical order among the geometries of his times. This was an outstanding achievement because from the late 18th century until the end of the 19th century approximately, geometry literally exploded and flourished into a complex and apparently disconnected tree. Klein’s approach profoundly influenced the mathematical development of geometry and it is now the standard accepted view.

Now, we turn to our examples.

- In Klein’s approach Euclidean geometry is the study of properties of figures in the plane $\mathbb{R}^2$ which are invariant under the rigid motions, that is, the isometries of $\mathbb{R}^2$ when it is equipped with the usual Euclidean metric d_2. Since, according to Klein’s view, the geometrical content is captured by the corresponding transformation group, it is an important problem to explore the structure of all transformations belonging to that group. In the present case, the problem is that how one can describe the structure of all isometries of $\mathbb{R}^2$. It is a remarkable result that every such transformation can be obtained from rotation, reflection and translation. Formulating the result in a more precise way, we have the following. If $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a map which satisfies
$$d_2(\phi(x), \phi(y)) = d_2(x, y) \qquad (x, y \in \mathbb{R}^2)$$
(i.e., it preserves the Euclidean distance), then ϕ is of the form
$$\phi(x) = Ux + a \qquad (x \in \mathbb{R}^2)$$
where U is a 2×2 orthogonal matrix and $a \in \mathbb{R}^2$ is a fixed vector. This result is valid in higher dimensions as well and what concerns infinite dimensions, we refer to the famous Mazur-Ulam theorem [157] stating that every surjective isometry between real normed spaces is affine. (According to a result of Charzyński, in finite dimensions the assumption of surjectivity can be omitted; see [49] or [20].)
One should observe the analogy (howsoever weak it is) between the present problem and the ones which belong to the class ‘Problem I’.
- Our second example concerns affine plane geometry. As the underlying space we take $\mathbb{R}^2$ as above. In Euclidean geometry, the notion of distance

is fundamental: circles with different radii are different. There are, however, geometric properties which are independent of metric information. Circles have properties as circles, not as circles of a certain radius. The same is true for all other geometric figures. Developing geometry with these requirements in mind leads to affine geometry. The relevant transformations of the plane in this case are the collinearities: a collinearity is a bijective map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the condition that for all triples x, z, y of distinct points, x, y, z are collinear if and only if $\phi(x), \phi(y), \phi(z)$ are collinear. It is a natural question that how one can describe those transformations in a more explicite way. The answer is given in a famous theorem due to Darboux. It states that the bijective map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a collinearity if and only if it is of the form

$$\phi(x) = Ax + b \qquad (x \in \mathbb{R}^2),$$

where A is an invertible 2×2 matrix and b is a fixed vector in $\mathbb{R}^2$.
Sometimes, Darboux theorem is formulated in a different way. Namely, one can say that the bijective map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ maps lines to lines if and only if it is of the form above.
One should remark that the first formulation of the theorem is in a quite close relation with Problem III while the second one is in some looser connection with Problem II.

- Our next example comes from projective geometry. One of the several forms of the fundamental theorem of projective geometry reads as follows. Let X be a linear space over a field and denote by $\mathcal{L}$ the set of all one-dimensional subspaces of X. A projectivity is a bijective map $\phi : \mathcal{L} \to \mathcal{L}$ which maps any three coplanar elements of $\mathcal{L}$ to coplanar elements. In other words ϕ has the property that whenever $e, f, g \in \mathcal{L}$ are such that $e \subset f + g$, we have $\phi(e) \subset \phi(f) + \phi(g)$. Now, the theorem states that if $\dim X \geq 3$ then a bijective map $\phi : \mathcal{L} \to \mathcal{L}$ is a projectivity if and only if it is of the form

$$\phi(e) = Se \qquad (e \in \mathcal{L})$$

where $S : X \to X$ is a bijective semilinear operator.
It is clear that the projectivities are defined as maps which preserve the relation 'coplanarity' and hence they can be considered as preservers in a sense similar to what appears in Problem III.

- Observe that the group automorphisms can be also viewed as preservers in some general sense. In fact, roughly speaking, these transformations are the bijective maps on a given group which preserve the group operation in a sense similar to what we have seen in Problem IV. We mean that the group automorphisms are precisely the bijective maps which commute (in a certain sense) with the function of 'taking products'. Of course, this problem can not be classified directly into the group 'Problem IV', but we feel unquestionable that it is of similar spirit. If this argument is accepted,

then going further on the way one can say that the isomorphisms between rings, algebras, etc. are also preservers in certain sense.
Regarding the automorphisms as preservers, in the particular cases we would like to know whether they are all of some nice form. To show one such case, we recall a beautiful result of Schreier and van der Waerden [222] concerning the form of all automorphisms of the general linear group. This group, that is the group $GL_n(\mathbb{F})$ of all nonsingular $n \times n$ matrices over the field $\mathbb{F}$ is a distinguished algebraic structure because of many reasons. Here we are interested in it because in the case when $n = 2$ and the underlying field is $\mathbb{R}$, it is the transformation group which represents 'normalized' affine plane geometry. Namely, it is the transformation group with collinearity and origin position as invariants. In Klein's approach the algebraic study of transformation groups is of fundamental importance. Therefore, it is important to explore the structure of the automorphisms of those groups. As for the present case, the corresponding result of Schreier and van der Waerden reads as follows. Let ϕ be an automorphism of $GL_n(\mathbb{F})$. Then there is a homomorphism $\gamma : GL_n(\mathbb{F}) \to \mathbb{F} \setminus \{0\}$, a field automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an element $T \in GL_n(\mathbb{F})$ such that ϕ is either of the form

$$\phi(A) = \gamma(A)T\tau(A)T^{-1} \qquad (A \in GL_n(\mathbb{F}))$$

or of the form

$$\phi(A) = \gamma(A)T\tau({A^{-1}}^{tr})T^{-1} \qquad (A \in GL_n(\mathbb{F})).$$

(Here $\tau(A)$ is the matrix which one obtains when applying τ for every entry of A.)

With the above list of examples we have hopefully given some evidence that preserver problems in the general sense appear in many parts of mathematics and hence the study of them certainly deserves attention.

In this book we present a number of our results concerning preserver problems in both the original strict sense and the extended sense as well. Moreover, we collect several of our results on the so-called local automorphisms of operator algebras in the study of which the 'preserver approach' has proved to be very useful.

0.2 Survey of the Results of Chapter 1

In Chapter 1 we collect some of our results concerning linear and multiplicative preserver problems on operator algebras and on function algebras.

In Section 1.1 we consider linear transformations on operator algebras which preserve operators having infinite rank and infinite corank. As we have already mentioned, preserver problems which concern rank play distinguished role among LPPs and a lot of work has been done on them both in the finite

and in the infinite dimensional cases. To mention only some of the corresponding papers, we refer to [17, 21, 66, 140, 246] for the finite dimensional case and to [58, 104, 203] for the infinite dimensional case. In our paper [90] we considered the problem of linear maps on operator algebras which preserve operators of a fixed corank. This problem is in some sense complementary to the problem of rank preservers. In Section 1.1 we study linear maps which are 'between' (finite) rank preservers and (finite) corank preservers, i.e., linear maps which preserve operators of infinite rank and infinite corank in both directions. It turns out immediately that, in general, transformations of this kind can be quite 'irregular': most of them can not be written in any of the nice compact forms that we have got used to when dealing with LPPs. Therefore, we restrict our attention to certain important subsets of operators with infinite rank and infinite corank such as the corresponding sets of idempotents (Theorem 1.1.1), projections (Theorem 1.1.2), and partial isometries (Theorem 1.1.3). We shall see that in those cases the corresponding preservers are all of one of the forms $X \mapsto AXB$, $X \mapsto AX^{tr}B$ with suitable operators A, B. In the last result of that section (Theorem 1.1.4) we describe the structure of the linear bijections of $B(H)$ which preserve left ideals in both directions. We prove that every such transformation is a two-sided multiplication corresponding to invertible operators.

In Section 1.2 we determine the structure of all linear maps on a von Neumann factor which preserve the extreme points of the unit ball. The study of this problem is motivated among others by a celebrated result of Kadison characterizing the surjective linear isometries of C^*-algebras. Clearly, every such isometry preserves the extreme points of the unit ball. In the main results of this section (Theorem 1.2.1 and Theorem 1.2.4) we see that on von Neumann factors the structure of all linear maps (not only the surjective ones!) which preserve the extreme points of the unit ball can be characterized in a way very similar to what Kadison obtained for the surjective isometries. For example, Theorem 1.2.1 shows that on an infinite factor each of our preservers can be written as a unitary multiple of a unital *-homomorphism or *-antihomomorphism.

In Section 1.3 we consider a preserver problem on the function algebra $C(X)$ of all complex valued continuous functions on the compact Hausdorff space X. The famous Banach-Stone theorem (see Theorem A.10) describes the general form of all surjective linear isometries of $C(X)$ with respect to the sup-norm. But beside the sup-norm, one can find it also natural to 'measure' a function with the diameter of its range. In view of the Banach-Stone theorem it is an immediate question that what are the linear bijections of $C(X)$ which preserve this quantity. In Theorem 1.3.1 we obtain that if X is first countable then our preservers are of a nice form. In fact, we prove that every linear bijection of $C(X)$ which preserves the diameter of the range of the functions $f \in C(X)$ is induced by a fixed homeomorphism of the underlying space X, a fixed rotation on $\mathbb{C}$ and translations in the vector space $C(X)$ with constant functions which depend linearly on f.

In Section 1.4 we present a result which can be viewed as a solution of a so-called multiplicative preserver problem. Such problems, that is, preserver problems concerning (not linear but) multiplicative maps on matrix algebras or, more generally, on operator algebras were first considered by Hochwald in [101]. (As recent papers, we refer to [52, 83] and [4] concerning the finite and infinite dimensional case, respectively.) In this section we study the *-semigroup endomorphisms of the operator algebra $B(H)$, that is, the multiplicative maps on $B(H)$ which preserve the adjoint in the sense as in Problem IV above. In Theorem 1.4.1 we show that if H is separable and infinite dimensional, then every such transformation satisfying a certain continuity condition (namely, having the Lipschitz property on commutative C^*-subalgebras) can be written in a nice form: it is a direct sum of the constant 0 map, the constant I map, some (unspecified number of) copies of the identity $A \mapsto A$, and some copies of the map $A \mapsto A^{*tr}$. We emphasize that we do not assume linearity. In fact, if the *-semigroup endomorphism under consideration is linear, then the problem can be reformulated asking that what are the *-representations of the operator algebra $B(H)$ on a separable Hilbert space. The answer to this question is well-known. Namely, by some deep results in the representation theory of $B(H)$ culminating in [121, 10.4.14 Corollary] we have that every such representation is the direct sum of some copies of the identity $A \mapsto A$.

0.3 Some Structures of Linear Operators in Quantum Mechanics

In the Hilbert space formulation of quantum mechanics, which is mainly due to von Neumann, several mathematical objects appear whose physical meaning is connected with the probabilistic aspects of the theory. The corresponding objects in which we are interested in the present work are the following. Let H be a (preferably complex) Hilbert space attached to a given quantum system.

(i) $S(H)$ denotes the set of all positive trace-class operators on H with trace 1. The elements of $S(H)$ are called the (normal) states of the system.
(ii) The extreme points of $S(H)$ as a convex set in $B(H)$ are called pure states. It is easy to see that they are exactly the rank-one projections on H. Their set is denoted by $P_1(H)$.
(iii) $P(H)$ stands for the set of all projections on H. This is an orthomodular lattice which describes the logical structure of quantum theory. In fact, the elements of $P(H)$ represent quantum events.
(iv) $B_s(H)$ denotes the set of all self-adjoint bounded linear operators on H. The elements of this set correspond to observable physical quantities.
(v) $E(H)$ stands for the set of all positive bounded linear operators on H which are majorized by the identity. The elements of $E(H)$ are called (Hilbert space) effects. They describe yes-no measurements which can be unsharp.

These sets are equipped with certain scalar valued functions and/or algebraic operations and/or binary relations which all have important physical content. The corresponding automorphisms, that is, the bijective maps on those sets which preserve the relevant structure represent different kinds of quantum mechanical symmetries. To describe those symmetries and to study the relations among them are important problems which were considered by a number of mathematicians and theoretical physicists. In what follows we briefly discuss the most well-known results obtained so far (see Chapter 2 of the recent book [47]).

We begin with Wigner's classical theorem on symmetry transformations [250, pp. 251-254] which concerns the set of all pure states equipped with the notion of transition probability. As written above, the set of pure states coincides with the set $P_1(H)$ of all rank-one projections on the Hilbert space H. A rank-one projection can be trivially identified with its range (which is a one-dimensional subspace) or with any unit vector which generates its range. Hence, one can regard pure states in (at least) three different ways: rank-one projections, one-dimensional subspaces, unit vectors (in this latter case the identification is one-to-one only up to multiplication by a scalar of modulus 1). We turn to the concept of transition probability. If $P, Q \in P_1(H)$ are pure states, then the transition probability between them is defined by $\operatorname{tr} PQ$, where tr stands for the usual trace functional. The mathematical meaning of this quantity can be better expressed by means of one-dimensional subspaces or unit vectors. In fact, it is easy to see that $\operatorname{tr} PQ$ is equal to $\cos^2\theta$, where θ is the angle between the ranges of P and Q as one-dimensional subspaces. Moreover, $\operatorname{tr} PQ$ is also equal to $|\langle\varphi, \psi\rangle|^2$, the square of the absolute value of the inner product of φ and ψ which are arbitrary representing unit vectors from the ranges of P and Q, respectively.

According to the above three different representations (we mean (a) rank-one projections and the trace of their product, (b) one-dimensional subspaces and the angle between them, (c) unit vectors and the absolute value of their inner product), we have three different but essentially equivalent concepts of symmetry transformations. Here, we mention only that one which concerns maps on $P_1(H)$. A bijective map $\phi : P_1(H) \to P_1(H)$ is called a symmetry transformation if it preserves the transition probability, i.e., if it satisfies

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in P_1(H)).$$

In this language, Wigner classical result reads as follows. (We shall meet other formulations of Wigner's theorem later.)

Wigner's theorem. *The bijective map $\phi : P_1(H) \to P_1(H)$ is a symmetry transformation if and only if there is an either unitary or antiunitary operator U on H such that*

$$\phi(P) = UPU^* \qquad (P \in P_1(H)). \tag{0.3.1}$$

The operator U is unique up to multiplication by a scalar of modulus 1.

In other words, the result says that any automorphism of $P_1(H)$ with respect to the structure induced by the transition probability arises from a semi-automorphism of the underlying Hilbert space (here we use the word 'semi' to designate that U can also be conjugate-linear). Moreover, the operator algebraic reformulation of this classical result tells us that any symmetry transformation of $P_1(H)$ extends to a *-automorphism or to a *-antiautomorphism of the C^*-algebra $B(H)$.

There is a nice and important generalization of Wigner's theorem due to Uhlhorn [237]. His result in the language what we have used to formulate Wigner's theorem reads as follows.

Uhlhorn's theorem. *Suppose that* $\dim H \geq 3$*. Let* $\phi : P_1(H) \to P_1(H)$ *be a bijective transformation which preserves the orthogonality between the elements of* $P_1(H)$ *in both directions, i.e., assume that* ϕ *has the property*

$$PQ = 0 \Longleftrightarrow \phi(P)\phi(Q) = 0 \qquad (P, Q \in P_1(H)).$$

Then ϕ *is of the form* (0.3.1) *with some unitary or antiunitary operator* U *on* H*.*

It is easy to see that $\operatorname{tr} PQ = 0$ is equivalent to $PQ = 0$. Hence, this result is really stronger than Wigner's original theorem as this one requires merely the preservation of zero transition probability. One can say that Uhlhorn's transformations preserve only the logical structure of a quantum mechanical system while Wigner's transformations preserve its complete probabilistic structure. The point is that in the case when $\dim H \geq 3$, Uhlhorn was able to obtain the same conclusion as Wigner.

These are the most fundamental results concerning the symmetries of the set of pure states.

As for the set $S(H)$ of all states, we have already mentioned that it is a convex set. The operation of convex combination on $S(H)$ is sometimes called mixture. The corresponding automorphisms are called mixture automorphisms (or affine automorphisms or S-automorphisms in the terminology of [45] or Kadison automorphisms in the terminology of [232]). Clearly, the mixture automorphisms are simply the bijective affine maps of $S(H)$. These are the most common automorphisms of the set of states. It is well-known (see e.g. [45, 232]) that the map $\phi : S(H) \to S(H)$ is a mixture automorphism if and only if it is of the form

$$\phi(A) = UAU^* \qquad (A \in S(H))$$

for some unitary or antiunitary operator U on H.

By an automorphism of the set $P(H)$ of all projections one usually means an ortho-order automorphism which is a bijective map $\phi : P(H) \to P(H)$ that preserves the order $\leq$ in both directions (this is just the usual order among self-adjoint operators) and the operation $\perp$: $P \mapsto I - P$ of orthocomplementation (which means that $\phi(P^\perp) = \phi(P)^\perp$ holds for every P). It follows from the fundamental theorem of projective geometry that, in case $\dim H \geq 3$, any ortho-order automorphism ϕ of $P(H)$ is of the form

$$\phi(P) = UPU^* \qquad (P \in P(H))$$

for some unitary or antiunitary operator U on H (see [45, 243]; a stronger result can be found in [71]).

Next consider the set $B_s(H)$ of all self-adjoint bounded linear operators (or, in other words, bounded observables) on H. This set is usually equipped with the operations of addition, multiplication by real scalars, and the so-called Jordan product $(A, B) \mapsto (AB + BA)/2$. The obtained structure is a real Jordan algebra. The most important automorphisms of $B_s(H)$ are the bijective maps $\phi : B_s(H) \to B_s(H)$ which preserve these operations. They are usually called Jordan automorphisms (or Segal automorphisms in the terminology of [232]). It is well-known that every such automorphism ϕ is implemented by an either unitary or antiunitary operator U on H which means that ϕ is of the form

$$\phi(A) = UAU^* \qquad (A \in B_s(H))$$

(see [45, 232]).

The notion of quantum effects is also a basic concept in the foundational studies of quantum physics [38, 63, 69, 130, 146]. These objects correspond to yes-no measurements that can be unsharp. In the Hilbert space formalism, quantum effects are represented by operators A on a Hilbert space H which satisfy $0 \leq A \leq I$. The set of all such operators on H (which are also called Hilbert space effects) is denoted by $E(H)$.

There are several relations and operations usually considered on $E(H)$ which all have serious physical meaning. We briefly summarize the most important ones.

(E1) There is a partial addition $\oplus$ on $E(H)$ which is defined in the following way. If $A, B \in E(H)$ are such that $A + B$ (the usual sum of operators) belongs to $E(H)$, then $A \oplus B := A + B$.

(E2) The usual ordering among self-adjoint operators gives rise to a partial order on $E(H)$. This is denoted by $\leq$. The map $\perp$: $A \mapsto A^\perp = I - A$ defines a kind of orthocomplementation on $E(H)$.

(E3) The set $E(H)$ is a convex subset of the linear space $B_s(H)$. Any convex combination $\lambda A + (1 - \lambda)B$ of effects A, B is called a mixture.

(E4) There is a kind of product called sequential product on $E(H)$ which is defined in the following way. If $A, B \in E(H)$, then their sequential product is $A \circ B := A^{1/2}BA^{1/2}$ which is easily seen to be an effect again.

All of the mentioned operations and relation are important because of the role they play in different aspects of quantum theory. Clearly, according to the above four items we obtain four different algebraic structures on the same set $E(H)$. Hence, we have four different kinds of automorphisms of $E(H)$.

(EA1) The automorphisms of $E(H)$ which correspond to the partial addition appearing in (E1) are called E-automorphisms [45] (or effect automorphisms [146]). Therefore, a bijective map $\phi : E(H) \to E(H)$ is an

E-automorphism if for any $A, B \in E(H)$ it satisfies

$$A + B \in E(H) \Longleftrightarrow \phi(A) + \phi(B) \in E(H)$$

and, in this case, we have

$$\phi(A + B) = \phi(A) + \phi(B).$$

(EA2) The automorphisms of $E(H)$ with respect to the partial order $\leq$ and the orthocomplementation $\perp$ are nowadays called ortho-order automorphisms (they were called *-automorphism in [146]). So, a bijective map $\phi : E(H) \to E(H)$ is an ortho-order automorphism if for any $A, B \in E(H)$ we

$$A \leq B \Longleftrightarrow \phi(A) \leq \phi(B)$$

and

$$\phi(A^{\perp}) = \phi(I - A) = I - \phi(A) = \phi(A)^{\perp}.$$

(EA3) The automorphisms of $E(H)$ corresponding to the operation of mixtures are called mixture automorphisms. Therefore, a bijective map $\phi : E(H) \to E(H)$ is a mixture automorphism if it is affine, i.e., satisfies

$$\phi(\lambda A + (1 - \lambda)B) = \lambda\phi(A) + (1 - \lambda)\phi(B)$$

for all $A, B \in E(H)$ and $\lambda \in [0, 1]$.

(EA4) The automorphisms of $E(H)$ with respect to the sequential product $\circ$ are called sequential automorphisms [79]. Trivially, a bijective map $\phi : E(H) \to E(H)$ is a sequential automorphism if

$$\phi(A \circ B) = \phi(A) \circ \phi(B)$$

holds for all $A, B \in E(H)$.

Just as with any algebraic structure in mathematics in general, it is an important question to explore and determine (if possible) the form of those automorphisms. Moreover, as they represent different kinds of quantum mechanical symmetries, to know their explicit forms is important also because of the applications.

Fortunately, we do know the structure of those automorphisms very well. Namely, any E-automorphism ϕ of $E(H)$ is of the form

$$\phi(A) = UAU^* \qquad (A \in E(H)) \tag{0.3.2}$$

where U is an either unitary or antiunitary operator [45, 146]. The ortho-order automorphisms are of the same form in the case when $\dim H \geq 2$. In fact, if $\dim H \geq 3$, this was proved by Ludwig in [146, Section V.5] (although the proof presented there contained some gaps which were recently clarified in [46]). In our paper [190] we showed that the same form applies also in the two-dimensional case and this solved a problem that was open for several

years. As for the mixture automorphisms of $E(H)$, they were determined in our paper [170]. It turned out that any such automorphism ϕ is either of the form

$$\phi(A) = UAU^* \qquad (A \in E(H)),$$

or of the form

$$\phi(A) = U(I - A)U^* \qquad (A \in E(H)),$$

where U is a unitary or antiunitary operator on H. Finally, what concerns the sequential automorphisms of $E(H)$, as an easy corollary of one of our results in [170], in their paper [79] Gudder and Greechie obtained that if $\dim H \geq 3$, then every sequential automorphism of $E(H)$ is of the form (0.3.2).

In Chapter 2 we collect some of our results on the above mentioned automorphisms and on certain preservers defined on different quantum structures.

0.4 Survey of the Results of Chapter 2

In Sections 2.1 and 2.2 we present generalizations of Wigner's and Uhlhorn's theorems. In order to formulate our results we have to continue the discussion about symmetry transformations that has been begun in the previous section. As we have mentioned there, there are several ways to define symmetry transformations which are essentially equivalent to each other.

Above we have said that a symmetry transformation is a bijective map on the set $P_1(H)$ of all pure states which preserves the transition probability (i.e., the trace of the product of rank-one projections). However, historically pure states were first viewed as rays in the Hilbert space H. If we have a unit vector $x \in H$, then the ray $\underline{x}$ corresponding to x is the set of all nonzero scalar multiples of x. If $x, y \in H$ are unit vectors, then we define

$$|\langle \underline{x}, \underline{y} \rangle| = |\langle x, y \rangle|.$$

(Recall that if P, Q are the rank-one projections onto the one-dimensional subspaces generated by x and y, respectively, then the above quantity equals the square-root of $\operatorname{tr} PQ$.) In this representation a symmetry transformation is a bijective map T on the set of all rays in H which satisfies

$$|\langle T\underline{x}, T\underline{y} \rangle| = |\langle \underline{x}, \underline{y} \rangle|$$

for every $\underline{x}, \underline{y}$. In this language, Wigner's theorem states that every symmetry transformation is induced by an either unitary or antiunitary operator on H in the following way. If T is a symmetry transformation, then there is a unitary or antiunitary operator U on H such that

$$T\underline{x} = \underline{Ux} \tag{0.4.1}$$

holds for every unit vector $x \in H$. It belongs to the history of Wigner's theorem that in the book [250] which appeared in 1931 Wigner did not give

a rigorous mathematical proof of his result. In fact, the first such proofs were published only in the 60's in the papers [14, 135]. The common feature of those proofs and most of the further ones (see, for example, [230, 212, 45] ordered chronologically) is that they manipulate in the underlying Hilbert space. This is not surprising since the problem is to 'linearize' the ray transformation T somehow. Therefore, such approaches are very natural on the one hand, but unfortunately the so-obtained proofs are quite lengthy and sometimes hard to follow. Just to imagine: in the language of vectors, the result says that if T is a symmetry transformation, then for every nonzero vector x we can choose a vector x' from the ray $T(\underline{x/\|x\|})$ with norm $\|x'\| = \|x\|$ such that the correspondence $x \mapsto x'$ is either linear or conjugate-linear. Obviously, we have the freedom to change the phase (i.e., we can multiply the vectors by any complex number of modulus 1), but this is all what we have. So, we start with some vector and move from one vector to another probably changing phases in every step in such a way that at the end the obtained transformation must be linear or conjugate-linear.

Instead of this and similar quite constructive but, in our eyes, very complicated arguments, in our paper [160] we invented a completely different approach to attack the problem. In fact, our idea proved to be very fruitful since it allowed us not only to give a short and easily understandable proof of Wigner's original result but also to generalize Wigner's theorem for different structures where the first mentioned approach would certainly break down. Our corresponding papers are [164, 166, 169, 171, 172, 174, 183].

The idea of our proof which is algebraic in character can be described in few sentences as follows. We consider a symmetry transformation ϕ on $P_1(H)$. For any finite system $P_1, P_2, \dots, P_n$ of (not necessarily orthogonal) rank-one projections and any corresponding system $\lambda_1, \lambda_2, \dots, \lambda_n$ of real numbers we define

$$\Phi(\sum_k \lambda_k P_k) = \sum_k \lambda_k \phi(P_k).$$

As ϕ preserves the transition probability, we can easily show that Φ is a well-defined real-linear transformation on the set $F_s(H)$ of all finite rank self-adjoint bounded linear operators on H. Using the properties of ϕ we deduce that Φ preserves the rank-one projections and the orthogonality between them. By the spectral theorem it follows readily that Φ has the property that

$$\Phi(A^2) = \Phi(A)^2 \qquad (A \in F_s(H)).$$

Next, we extend Φ to whole set $F(H)$ of finite rank (bounded linear) operators on H by the formula

$$\tilde{\Phi}(A + iB) = \Phi(A) + i\Phi(B) \qquad (A, B \in F_s(H)).$$

It is easy to check that $\tilde{\Phi}$ is a linear transformation on $F(H)$ which preserves squares, i.e., it has the property that

$$\tilde{\Phi}(T^2) = \tilde{\Phi}(T)^2 \qquad (T \in F(H)).$$

This means that $\tilde{\Phi}$ is a Jordan homomorphism of $F(H)$. As ϕ is bijective, it can be seen that $\tilde{\Phi}$ is also bijective. Now, it follows from the well-known result Theorem A.7 of Herstein [100] that $\tilde{\Phi}$ is either an automorphism or an antiautomorphism of $F(H)$. But the form of those transformations is described in Theorem A.8. The proof can now be completed very easily. For details see [160] or [172, Remark] and we also refer to the proof of Theorem 2.1.1 in the present work.

The above described operator algebraic approach made possible for us to generalize Wigner's original theorem in several directions. In Section 2.1 we present such a result which has interesting geometrical content. Namely, we prove a Wigner-type result for transformations defined on the set of all n-dimensional subspaces of H (n is a fixed positive integer) which preserve the so-called principal angles. To formulate the result observe that one can state Wigner's original theorem also in the following way: Any bijective transformation T on the set of all one-dimensional subspaces of a Hilbert space H which preserves the angles between those subspaces is induced by either a unitary or an antiunitary operator on H. More precisely, for any such T there is an either unitary or antiunitary operator U on H for which

$$T(L) = U[L] = \{Ux \,:\, x \in L\} \tag{0.4.2}$$

holds for every one-dimensional subspace L. The main result Theorem 2.1.1 in Section 2.1 states that every transformation (not necessarily bijective) on the set of all n-dimensional subspaces of a real or complex Hilbert space which preserves the principal angles is induced by a linear or conjugate-linear isometry of H in the same way as in (0.4.2).

In Section 2.2 we present a generalization of Uhlhorn's version of Wigner's theorem concerning ray transformations on indefinite inner product spaces. Our corresponding result significantly generalizes a result of Van den Broek. In fact, in [32] he obtained an Uhlhorn-type result for indefinite inner product spaces which are induced by nonsingular self-adjoint operators on a finite (at least 3-)dimensional complex Hilbert space. Our result is a far-reaching extension of Van den Broek's theorem since it holds for both real and complex Hilbert spaces of any dimension (not less than 3) and indefinite inner products induced by arbitrary nonsingular operators.

The proof of this result of ours is again operator algebraic in character. In fact, the main result Theorem 2.2.1 of that section (from which we can deduce our Uhlhorn-type result) gives a complete description of the bijective transformations ϕ on the set of all rank-one idempotents on a given real or complex Banach space X with $\dim X \geq 3$ which preserve zero product in both directions. For example, in the infinite dimensional case the result asserts that every such transformation ϕ extends to a linear or conjugate-linear algebra automorphism of $B(X)$, i.e., there is an invertible bounded linear or conjugate-linear operator $A \in B(X)$ such that ϕ is of the form

$$\phi(P) = APA^{-1}$$

for every rank-one idempotent P on X. Using this theorem we can prove our Uhlhorn-type result Corollary 2.2.3 concerning ray transformations on indefinite inner product spaces. It turns out that under the conditions what we have mentioned in the previous paragraph, i.e., given a real or complex Hilbert space H with $\dim H \geq 3$ and an arbitrary nonsingular operator on H, for any ray transformation T on the induced indefinite inner product space which preserves orthogonality in both directions there exists an invertible either linear or conjugate-linear operator U on H such that

$$T\underline{x} = \underline{Ux}$$

for every nonzero $x \in H$.

In Sections 2.3 and 2.4 we present the descriptions of all bijective transformations of the set $S(H)$ of all states which preserve a certain given function operating on pairs of states.

The results in Section 2.3 concern those bijective transformations of $S(H)$ which preserve the so-called fidelity. If A, B are states (or, more generally, positive trace-class operators), then the fidelity between them is defined by

$$F(A, B) = \operatorname{tr}(A^{1/2}BA^{1/2})^{1/2}.$$

The concept in this form was introduced by Uhlmann in [241] but one should consult also the papers [116] and [238]. Nowadays this notion plays very important role in several extensive research areas in quantum theory like quantum computation and quantum information theory. Fidelity can be viewed as the natural extension of the notion of transition probability from the case of pure states to the case of all states. In fact, if P, Q are rank-one projections, then it is easy to see that we have

$$F(P, Q) = (\operatorname{tr} PQ)^{1/2},$$

i.e., $F(P, Q)$ is equal to the square-root of the transition probability between P and Q. Keeping Wigner's fundamental result in mind, it is a natural question that what are the bijective transformations on $S(H)$ which preserve the fidelity. This problem was raised by Uhlmann. The answer is presented in Theorem 2.3.2 which tells us that every such transformation is implemented by an either unitary or antiunitary operator. This is the main result in Section 2.3.

Motivated by certain problems in physics, in many cases the set of all states is considered as a metric space. In fact, there are a number of metrics on $S(H)$ which fit to different physical problems. However, as one can learn from [93], the most important such metrics can be derived from two fundamental ones, namely, from the Bures metric and from the one which is induced by the trace-norm. (We note that the Bures metric is in an intimate connection with

Uhlmann's fidelity as we shall see.) Since the corresponding isometries represent certain kinds of symmetries, Uhlmann raised the problem to determine their forms. We give the solution in Section 2.4 . Namely, in Theorems 2.4.2 and 2.4.4 we shall see that all those isometries are implemented by unitary or antiunitary operators.

In Sections 2.5 and 2.6 we present some of our results concerning preservers on the set $B_s(H)$ of all bounded observables.

The main result of Section 2.5, namely Theorem 2.5.2 describes the general form of all bijective maps (no linearity is assumed) of $B_s(H)$ which preserve the usual order $\leq$ in both directions. Surprisingly enough, it turns out that in the case when $\dim H > 1$, every such transformation is automatically affine and of the form

$$A \mapsto TAT^* + X$$

with some invertible bounded either linear or conjugate-linear operator T on H and a fixed element $X \in B_s(H)$. Some corollaries of this result concerning bijective maps of $B_s(H)$ which preserve the order and other physically relevant relations (like compatibility or complementarity) are also presented.

In Section 2.6 we consider linear transformations on $B_s(H)$ which preserve important probabilistic characteristics of observables such as moments or variance. We show that each bijective linear transformation which preserves any of those quantities is 'almost' a Jordan automorphism of $B_s(H)$. For example, Theorem 2.6.2 (which is the main result of that section) states that if a bijective linear transformation $\phi : B_s(H) \to B_s(H)$ preserves the so-called maximal deviation of observables (that is, the supremum of the set of square-roots of the variances in all possible pure states; see (2.6.3)), then it is of the form

$$\phi(A) = \pm UAU^* + f(A)I,$$

where U is a unitary or antiunitary operator on H and f is a linear functional on $B_s(H)$. The solution of a non-linear version of the problem is also presented.

In Sections 2.7 and 2.8 we consider certain algebraic structures on the set of Hilbert space effects and study their transformations.

In Section 2.7 we deal with a certain class of preservers on $E(H)$. They are the bijective maps on $E(H)$ which preserve the order and zero product in both directions. In the main result of the section we determine the general form of all those transformations. Namely, Theorem 2.7.1 tells us that if $\dim H \geq 2$ and the bijective map $\phi : E(H) \to E(H)$ preserves the order and zero product in both directions, then there is an either unitary or antiunitary operator U on H and a real number $p < 1$ such that with the real function $f_p(x) = \frac{x}{xp+(1-p)}$ ($x \in [0,1]$) we have

$$\phi(A) = Uf_p(A)U^* \qquad (A \in E(H)).$$

(Here $f_p(A)$ denotes the image of the function f_p under the continuous function calculus belonging to the self-adjoint operator A.) In Corollary 2.7.3 we

obtain that the same nice form holds for any bijection of $E(H)$ which preserves the order in both directions and sends one single nontrivial scalar operator to an operator of the same type. In Corollary 2.7.5 we easily deduce that every bijective map $\phi : E(H) \to E(H)$ which preserves the order and the so-called coexistency in both directions is of the form

$$\phi(A) = UAU^* \qquad (A \in E(H)) \tag{0.4.3}$$

for some unitary or antiunitary operator U on H. In the last two results Corollary 2.7.6 and Corollary 2.7.7 we obtain that the ortho-order automorphisms and the sequential automorphisms of $E(H)$ are implemented by unitary or antiunitary operators just as in (0.4.3). In fact, we shall see that both of those two types of automorphisms preserve the order and zero product in both directions when they are defined on $E(H)$ and, moreover, also when their domain is the set all effects which belong to a given von Neumann algebra. Therefore, the bijective maps which preserve the order and zero product in both directions provide a common frame to study the ortho-order automorphisms and sequential automorphisms. It should also be emphasized that the results of the section hold for Hilbert spaces of dimension not less than 2. In the two-dimensional case we obtain extensions of several former results which were known before only for the case $\dim H \geq 3$.

After introducing the concept of effects in an arbitrary C^*-algebra, in Section 2.8 we present a description of the sequential isomorphisms between the sets of von Neumann algebra effects. The main result Theorem 2.8.1 of this section is a far-reaching generalization of a result of Gudder and Greechie [79] which gives the general form of sequential automorphisms of $E(H)$ under the condition that $\dim H \geq 3$. We shall see that in the particular case when the underlying von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ have no commutative direct summands, every sequential isomorphism $\phi : E(\mathcal{A}) \to E(\mathcal{B})$ between the sets of their effects extends to the direct sum of a linear *-isomorphism and a linear *-antiisomorphism which maps $\mathcal{A}$ onto $\mathcal{B}$.

0.5 Local Derivations, Local Automorphisms and Local Isometries of Operator Algebras and Function Algebras

In the last decades considerable work has been done on certain local maps of operator algebras. The main problem in this area of research is to answer the question whether the local actions of some important classes of transformations (like derivations, automorphisms, isometries) on a given operator algebra determine the class under consideration completely.

The originators of investigations of this kind are Kadison, Larson and Sourour. In [119] Kadison studied the local derivations of a von Neumann algebra $\mathcal{R}$. He called a continuous linear map $\delta : \mathcal{R} \to \mathcal{R}$ a local derivation if it coincides with some derivation at each point in the algebra (the derivations

possibly differing from point to point). More precisely, it is supposed that for every $a \in \mathcal{R}$ there exists a derivation δ_a on $\mathcal{R}$ such that $\delta(a) = \delta_a(a)$. (In fact, in [119] Kadison considered the more general case of dual $\mathcal{R}$-bimodule valued local derivations.) Kadison's investigation was motivated by some problems concerning the Hochschild cohomology of operator algebras. He proved in [119] that in the above setting, every local derivation is a (global) derivation. Independently and approximately at the same time, Larson and Sourour proved in [134] that similar conclusion holds true for the local derivations of the full operator algebra $B(X)$ on a Banach space X. (We note that in [134] local derivation means any linear map on $B(X)$ which pointwise equals a derivation, so continuity is not assumed there.)

Beside derivations, there are at least two additional very important classes of transformations on operator algebras which certainly deserve attention from the point of view described above. These are the group of automorphisms and the group of surjective linear isometries. The automorphism group reflects the algebraic properties of the underlying algebra while the isometry group reflects its geometrical structure. In [133, Some concluding remarks (5), p. 298], motivated by the extensive research on reflexive linear subspaces of $B(H)$, Larson initiated the study of local automorphisms of Banach algebras. The definition is straightforward: a local automorphism is a linear map ϕ on a given Banach algebra $\mathcal{A}$ with the property that for every $x \in \mathcal{A}$ there exists an automorphism ϕ_x of $\mathcal{A}$ such that $\phi(x) = \phi_x(x)$. In other words, ϕ pointwise equals an automorphism that may vary from point to point. In the paper [134], Larson and Sourour proved that if X is an infinite dimensional Banach space, then every surjective local automorphism of $B(X)$ is an automorphism (also see [25]). Afterwards, in their important paper [27] Brešar and Šemrl showed that in the case of a separable infinite dimensional Hilbert space H the above conclusion holds true without the assumption of surjectivity, i.e., every local automorphism of $B(H)$ is an automorphism.

In relation with the name of Larson, above we have mentioned the concept of reflexivity. In fact, one can regard local maps also from this perspective. We recall the notion of reflexivity in the sense of Loginov and Shulman [141]. If $\mathcal{S}$ is a linear subspace of the algebra $B(H)$ of all bounded linear operators on the Hilbert space H, then set

$$\operatorname{ref} \mathcal{S} = \{T \in B(H) \, : \, Tx \in \overline{\mathcal{S}x} \text{ for all } x \in H\}$$

(bar stands for the closure operation). We say that $\mathcal{S}$ is reflexive if $\operatorname{ref} \mathcal{S} = \mathcal{S}$. The study of reflexive linear subspaces is one of the main research areas in operator theory as it is intimately connected with the fundamental problem of invariant subspaces. A nice introduction to the subject can be found, for example, in the recent book [55] of Conway. (For an interesting general view of reflexivity we refer to the paper [91] of Hadwin.)

Obviously, in the above definition of reflexivity the assumptions that the underlying space is a Hilbert space and $\mathcal{S}$ is a linear subspace are in fact

not essential. Therefore, we can introduce the following notions. Let $\mathcal{X}$ be a Banach space and $\mathcal{E}$ an arbitrary subset of $B(\mathcal{X})$. Define

$$\operatorname{ref}_{al} \mathcal{E} = \{T \in B(\mathcal{X}) : Tx \in \mathcal{E}x \text{ for all } x \in \mathcal{X}\}$$

and

$$\operatorname{ref}_{to} \mathcal{E} = \{T \in B(\mathcal{X}) : Tx \in \overline{\mathcal{E}x} \text{ for all } x \in \mathcal{X}\}.$$

The sets $\operatorname{ref}_{al} \mathcal{E}$ and $\operatorname{ref}_{to} \mathcal{E}$ are called the algebraic reflexive closure of $\mathcal{E}$ and the topological reflexive closure of $\mathcal{E}$, respectively. The collection $\mathcal{E}$ of transformations is called algebraically reflexive if $\operatorname{ref}_{al} \mathcal{E} = \mathcal{E}$. Similarly, $\mathcal{E}$ is said to be topologically reflexive if $\operatorname{ref}_{to} \mathcal{E} = \mathcal{E}$.

With this terminology we can reformulate Kadison's result in [119] mentioned above saying that the set (or, more precisely, Lie algebra) of all derivations of a von Neumann algebra $\mathcal{R}$ is algebraically reflexive (in this case the Banach space $\mathcal{X}$ is equal to $\mathcal{R}$). Similarly, it follows from [134] that the same conclusion holds for the derivation algebra of $B(X)$, X being any Banach space (in this case we have $\mathcal{X} = B(X)$).

It is undoubtedly a remarkable fact on an operator algebra (or, more generally, on a Banach algebra) if the Lie algebra of all of its derivations or the group of all of its automorphisms is algebraically or topologically reflexive. In fact, this means that those collections of transformations are completely determined by their local actions (in the case of algebraic reflexivity) or by their approximate local actions (in the case of topological reflexivity). Roughly speaking, the algebraic reflexivity of a given collection of transformations means that if a continuous linear map pointwise belongs to the collection, then it globally belongs to it. Similarly, the topological reflexivity of a given collection of transformations means that if a continuous linear map pointwise approximately belongs to the collection, then it globally belongs to it.

The study of such questions issued in several important and interesting results. Beside the above mentioned papers we also refer

- to [23, 56, 92, 114, 200, 221, 249, 251, 252, 254] for results on local derivations and on the algebraic reflexivity of the derivation algebra,
- to [111, 112, 231] for results on the topological reflexivity of the derivation algebra,
- to [41, 88, 92, 109, 214, 216, 221] for results on local automorphisms and local isometries, and on the algebraic or topological reflexivity of the automorphism and isometry groups.

Our results on the algebraic or topological reflexivity of the automorphism and isometry groups of operator algebras and function algebras appeared in the papers [16, 161, 162, 163, 168, 188, 191, 196, 197] (also see [173, 192]).

Above we have considered local maps in the following sense. Given a linear algebraic structure $\mathcal{X}$ (mainly an operator algebra) and a collection $\mathcal{E}$ of its linear transformations, the corresponding local maps are the functions ϕ which have the following properties:

(a) ϕ is linear and
(b) for every $x \in \mathcal{X}$ there exists a transformation $\phi_x \in \mathcal{E}$ such that $\phi(x) = \phi_x(x)$.

The results we have referred to show that in many important cases our local maps are all 'global', i.e., they belong to the given class of transformations. All those results concern linear algebraic structures. Since they have proved to be nice and important achievements, it is a natural idea to extend the territory of such investigations from linear structures to more general ones. In order to do so, we clearly have to omit the condition (a) requiring linearity. But in that case it is quite apparent that the second property (b) alone is too weak to give reasonable results even on operator algebras. For example, if we consider any function ϕ on the operator algebra $B(H)$ such that for every $A \in B(H)$ the operator $\phi(A)$ belongs to the similarity orbit of A, then ϕ pointwise belongs to the automorphism group of $B(H)$ but in general ϕ fails to be an automorphism. So, if we would like to omit the condition (a), we have to strengthen the condition (b).

One simple idea leads to the concept of 2-locality. In fact, in his paper [224] Šemrl introduced the following definition. Let $\mathcal{A}$ be an algebra. The transformation $\phi : \mathcal{A} \to \mathcal{A}$ (linearity is not assumed) is called a 2-local automorphism if for every $x, y \in \mathcal{A}$, there is an automorphism $\phi_{x,y}$ of $\mathcal{A}$ for which $\phi(x) = \phi_{x,y}(x)$ and $\phi(y) = \phi_{x,y}(y)$. The definition of 2-local derivations is similar. The main results of [224] show that if H is an infinite dimensional separable Hilbert space, then every 2-local automorphism of $B(H)$ is an automorphism and similar assertion holds concerning the 2-local derivations. These nice and highly nontrivial results justify the usefulness of Šemrl's definition. Clearly, the above notion of 2-local automorphisms has the great advantage that it can be defined on arbitrary algebraic structures not only on algebras and this is what we have been looking for. Now one can define 2-local maps of different kinds and study them on different structures. Obviously, the main problem to answer should be the following question: Given an algebraic structure and a class of transformations on it, is it true that the corresponding 2-local maps are all global?

This general concept of 2-locality is rather new and relatively few results have been obtained so far concerning it. We mention the papers [13, 57, 85, 126, 127, 253] and our corresponding works [179, 182, 184, 193]. Nevertheless, we believe that the study of 2-local automorphisms is an important problem because it can give valuable new information on the automorphism groups appearing in different parts of mathematics. Therefore, we expect that the attention paid to this kind of investigations and the intensity of research in this direction will increase in the near future.

Finally, we note that the basic problem of local transformations seems to have nothing to do with preserver problems. However, as it will turn out in the proofs of our results on local maps, we often use 'preserver arguments' as well as several particular preserver results. In this respect the area of local

automorphisms and isometries can be considered as a field of applications of preservers.

0.6 Survey of the Results of Chapter 3

In order to present the results of Chapter 3 we have to continue the discussion about former results concerning the reflexivity of certain classes of transformations on operator algebras.

There is an important and beautiful result of Shulman on the reflexivity of the derivation algebra. He proved in [231] that it is topologically reflexive in the case of any C^*-algebra. If such a general result holds for derivations, one can ask that what is the situation with the automorphisms. Unfortunately, this is a completely different story. In fact, there exists a (unital and commutative) C^*-algebra whose automorphism group is not reflexive even algebraically. An example is given in Remark 3.2.2. The absence of such a general result motivates us to consider important particular C^*-algebras. The most simple and fundamental such algebras are $B(H)$, the algebra of all bounded linear operators on a Hilbert space H and $C(X)$, the algebra of all continuous complex valued functions on a compact Hausdorff space X.

As for $B(H)$, we have already learnt from [27] that if H is infinite dimensional separable then the automorphism group of $B(H)$ is algebraically reflexive. (In the finite dimensional case the result is no longer valid, see Remark 3.1.7.) In Section 3.1 we show that even more is true. Namely, Theorems 3.1.2 and 3.1.3 tell us that the automorphism and isometry groups of $B(H)$ (H being infinite dimensional separable) are topologically reflexive. The proofs of these statements rest heavily on the rather surprising result Theorem 3.1.1 which asserts that if a Jordan homomorphism ϕ of $B(H)$ has the property that its range contains two extreme operators, one with rank one and one with dense range, then ϕ is automatically surjective and, furthermore, it is either an automorphism or an antiautomorphism of $B(H)$.

In Section 3.2 we consider the reflexivity problem of the automorphism and isometry groups for function algebras. There we deal only with the case of the most fundamental such algebra which is $C(X)$, the algebra of all continuous complex valued functions on a compact Hausdorff space X. As it follows from Remark 3.2.2, for general X we do not have even the algebraic reflexivity of the considered groups. Nevertheless, in the simple result Theorem 3.2.1 we obtain that if X is a first countable compact Hausdorff space then the automorphism and isometry groups of $C(X)$ are algebraically reflexive. (As for topological reflexivity, we note that it does not hold even when X is a compact interval of the real numbers. See Subsection 3.1.2.)

So, we have results concerning the topological reflexivity of the automorphism and isometry groups of $B(H)$ and a result concerning the algebraic reflexivity of the automorphism and isometry groups of $C(X)$. It is a natural question that what happens to reflexivity if we compose a new algebra from

two given ones with nice reflexivity properties. For example, one should be interested in taking tensor product. Unfortunately, as we shall see in Section 3.3, the problem in full generality seems to be completely hopeless. But in the case of the function algebra $C_0(X)$ of all continuous complex valued functions on a locally compact Hausdorff space vanishing at infinity and $B(H)$ (H being infinite dimensional separable) we have positive results. In fact, in Theorem 3.3.2 we prove that if the automorphism group of $C_0(X)$ is algebraically reflexive, then so is the automorphism group of the tensor product $C_0(X) \otimes B(H)$. As for the isometry group, we have a similar result in the case when the underlying topological space X is σ-compact. This is the content of Theorem 3.3.3. In Theorem 3.3.4 we show that the automorphism and isometry groups of $C_0(\mathbb{R}^n)$ are algebraically reflexive. As a consequence, in Corollary 3.3.5 we obtain that the same holds for the tensor product $C_0(\mathbb{R}) \otimes B(H)$ as well. The reason to emphasize this corollary is that tensor products of the form $C_0(\mathbb{R}) \otimes \mathcal{A}$ ($\mathcal{A}$ being an arbitrary C^*-algebra) are particularly important. In fact, $C_0(\mathbb{R}) \otimes \mathcal{A}$ is called the suspension of the C^*-algebra $\mathcal{A}$ and is usually denoted by $S\mathcal{A}$. This concept plays very important role in the K-theory of C^*-algebras since the K_1-group of $\mathcal{A}$ is well-known to be isomorphic to the K_0-group of $S\mathcal{A}$.

In Sections 3.4 and 3.5 we present some results of ours on 2-local automorphisms. As mentioned above, in his paper [224] Šemrl proved that for an infinite dimensional separable Hilbert space H, every 2-local automorphism of $B(H)$ is an automorphism. Moreover, he admitted that the same is true in the finite dimensional case and, as he could give only a long proof of it involving tedious computations, he raised the problem to present a shorter argument. In Section 3.4 we give such a short proof. In fact, in Corollary 3.4.2, respectively in Proposition 3.4.3 we prove that every 2-local automorphism of the algebra of all $n \times n$ matrices over an algebraically closed field of characteristic 0 or over the field of real numbers is an automorphism. Our approach to attack the problem also leads to a remarkable generalization of Šemrl's original result. Namely, in Theorem 3.4.4 we obtain the following result. If X is a real or complex Banach space with a Schauder basis and $\mathcal{A}$ is a subalgebra of $B(X)$ which contains all compact operators, then every 2-local automorphism of $\mathcal{A}$ is an automorphism. As most classical separable Banach spaces have Schauder bases, this is a substantial generalization of the result in [224].

In Section 3.5 we consider the 2-local automorphisms of some non-linear quantum structures. In Theorem 3.5.1 we prove that every continuous 2-local automorphism of the poset (partially ordered set) of all idempotents on an infinite dimensional separable Hilbert space is an automorphism. Theorem 3.5.2 tells us that similar assertion holds for the 2-local automorphisms of the orthomodular lattice of all projections of H even without assuming continuity. Finally, we obtain in Theorem 3.5.3 that every 2-local automorphism of the Jordan algebra $B_s(H)$ of all self-adjoint operators is an automorphism.

0.7 Notation

In this section we fix the notation and some definitions that we shall use (or have already used) frequently in our work.

Firstly, unless explicitly stated otherwise, every linear space which appears in the book is considered over the complex field $\mathbb{C}$.

For Chapters 1 and 3 we fix the following.

For an arbitrary positive integer n, $M_n(\mathbb{C})$ denotes the algebra of all $n \times n$ complex matrices. The transpose of a matrix $A \in M_n(\mathbb{C})$ is denoted by A^{tr}.

If X is a Banach space, then $B(X)$ stands for the algebra of all bounded linear operators on X. The ideals of all finite rank operators and all compact operators in $B(X)$ are denoted by $F(X)$ and $K(X)$, respectively. Any subalgebra of $B(X)$ which contains $F(X)$ is called a standard operator algebra on X.

If $A \in B(X)$, then $\ker A$ denotes the kernel of A while $\operatorname{rng} A$ stands for the range of A. The spectrum of A is denoted by $\sigma(A)$.

The operator $P \in B(X)$ is called an idempotent if $P^2 = P$.

Let H be a Hilbert space. The inner product on H is denoted by $\langle ., . \rangle$. If $x, y \in H$, then $x \otimes y$ stands for the operator defined by $(x \otimes y)z = \langle z, y \rangle x$ $(z \in H)$. If $A \in B(H)$, then its adjoint (in the Hilbert space sense) is denoted by A^*. Fixing an arbitrary complete orthonormal system in H and considering the corresponding matrix representation of operators, one can define the transpose A^{tr} of an arbitrary operator $A \in B(H)$ in the obvious way.

The operator $P \in B(H)$ is called a projection if it is a self-adjoint idempotent. If $U \in B(H)$ is a surjective isometry, then it is called unitary. By an antiunitary operator we mean a surjective conjugate-linear isometry on H. The operator $W \in B(H)$ is called a partial isometry if it is an isometry on a closed subspace M of H and is 0 on $M^\perp$, the orthogonal complement of M.

If X is a compact Hausdorff space, then $C(X)$ denotes the algebra of all continuous complex valued functions on X.

Let $\mathcal{A}$ be a C^*-algebra. If not explicitly stated otherwise, we automatically assume that $\mathcal{A}$ is unital.

Let $\mathcal{A}, \mathcal{B}$ be algebras. The linear transformation $\phi : \mathcal{A} \to \mathcal{B}$ is called a Jordan homomorphism if it satisfies

$$\phi(A^2) = \phi(A)^2 \qquad (A \in \mathcal{A}). \tag{0.7.1}$$

It is easy to see that this equality is equivalent to

$$\phi(AB + BA) = \phi(A)\phi(B) + \phi(B)\phi(A) \qquad (A, B \in \mathcal{A})$$

(just write $A + B$ in the place of A in (0.7.1) and compute). A linear transformation $\psi : \mathcal{A} \to \mathcal{B}$ is called a homomorphism if it satisfies

$$\psi(AB) = \psi(A)\psi(B) \qquad (A, B \in \mathcal{A})$$

while ψ is said to be an antihomomorphism if it satisfies

$$\psi(AB) = \psi(B)\psi(A) \qquad (A, B \in \mathcal{A}).$$

Obviously, homomorphisms and antihomomorphisms between algebras are examples for Jordan homomorphisms.

The meaning of the corresponding concepts like Jordan automorphism, Jordan isomorphism, etc. are supposed to be clear.

Let $\mathcal{A}, \mathcal{B}$ be *-algebras. A linear transformation $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a Jordan *-homomorphism if it is a Jordan homomorphism which preserves the adjoint operation, i.e., which satisfies

$$\phi(A^*) = \phi(A)^* \qquad (A \in \mathcal{A}).$$

The definition of a *-homomorphism or a *-antihomomorphism is analogous.

In addition to the above, in Chapter 2 we use the following notation and definitions.

The ideal of all trace-class operators in $B(H)$ is denoted by $C_1(H)$ and tr stands for the usual trace functional on it. The set of all positive elements in $C_1(H)$ which we call density operators is denoted by $C_1^+(H)$. The normalized elements of $C_1^+(H)$, i.e., the ones with trace 1 are called (normal) states and they form the set $S(H)$.

The set of all projections on H is denoted by $P(H)$ and $P_1(H)$ stands for the set of all rank-one elements in $P(H)$. The operators in $P_1(H)$ are also called pure states.

The set of all self-adjoint bounded linear operators on H is denoted by $B_s(H)$. The elements of this set are also called (bounded) observables as they represent observable physical quantities.

The set of all positive bounded linear operators on H which are bounded by the identity I (i.e., the set of all operators $A \in B(H)$ for which $0 \leq A \leq I$) is denoted by $E(H)$. The elements of $E(H)$ are called Hilbert space effects. They represent yes-no quantum measurements which can be unsharp.