Lecture Notes in Mathematics 1894

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Stable Approximate Evaluation of Unbounded Operators

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In Memory of Joaquin Bustoz, Jr.
1939-2003

Preface

This monograph is a study of an aspect of operator approximation theory that emerges from the theory of linear inverse problems in the mathematical sciences. Such inverse problems are often modeled by operator equations of the first kind involving a compact linear operator defined on a Hilbert space. The conventional solution operator for the inverse problem is the Moore-Penrose generalized inverse of the model operator. Except in the unusual case when the model operator has finite rank, the Moore-Penrose inverse is a densely defined, closed, unbounded operator. Therefore bounded perturbations in the data can be amplified without bound by the solution operator. Indeed, it is a common experience of those who have dealt with such inverse problems to find that low amplitude noise in the data expresses itself as high amplitude oscillations in the computed solution. The successful treatment of these inverse problems therefore requires two ingredients: an approximation of the Moore-Penrose inverse coupled with a stabilization technique to dampen spurious oscillations in the approximate solution.

Here we consider stabilized evaluation of an unbounded operator as a problem in its own right in operator approximation theory. By stabilized evaluation we mean that the value of an unbounded operator at some vector in its domain is approximated by applying *bounded* linear operators to an approximate data vector that is not necessarily in the domain of the original unbounded operator. Questions of convergence and orders of approximation will be of foremost concern. A unifying thread for the discussion is a classical theorem of von Neumann on certain bounded "resolvents" of closed densely defined unbounded operators. This result is the bridge that allows passage from the unbounded operator to a class of approximating bounded operators. When von Neumann's theorem is combined with the spectral theorem for bounded self-adjoint operators a general scheme for stabilized evaluation of the unbounded operator results. Particular cases of the general scheme, notably the Tikhonov-Morozov method and its variants, are studied in some detail and finite-dimensional realizations are dealt with in the final chapter.

The key idea of von Neumann's proof involves regarding the graph of the operator as a subspace embedded in a product Hilbert space (in this sense the proof is truly *Cartesian*). We also show that this notion, combined with von Neumann's alternating projection theorem, applied in the product space, can be used to give an alternate non-spectral proof of one of the best known operator stabilization methods.

Our intent is for the monograph to be reasonably self-contained. We begin with a fairly informal introductory chapter in which a number of model inverse problems leading to the evaluation of unbounded operators are introduced. The next chapter fills in background material from functional analysis and operator theory that is helpful in the sequel. We hope that this approach will make the monograph a useful source of collateral reading for students in graduate courses in functional analysis and related courses in analysis and applied mathematics.

Much of the work that is reported here was originally carried out in collaboration with my friends Otmar Scherzer of the University of Innsbruck, Austria and Martin Hanke-Bourgeois of the University of Mainz, Germany. With both Otmar and Martin I had the happy experience of open and friendly collaborations in which my benefits exceeded my contributions.

While writing these notes I learned of the tragic death of my earliest colleague and coauthor, Joaquin Bustoz, Jr., of Arizona State University. Besides his many research papers on summability theory and special functions, Joaquin's important and lasting contributions to the mathematical education of disadvantaged youth made his passing a great loss to the profession, to say nothing of the personal sense of loss felt by his friends and colleagues. This monograph is fondly dedicated to Joaquin's memory.

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Seabrook Island, S.C. *Charles Groetsch*

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