Lecture Notes in Mathematics 1831

Springer
Berlin
Heidelberg
New York
Hong Kong
London
Milan
Paris
Tokyo

A. Connes J. Cuntz E. Guentner
N. Higson J. Kaminker J. E. Roberts

Noncommutative Geometry

Lectures given at the
C.I.M.E. Summer School
held in Martina Franca, Italy,
September 3-9, 2000

Editors: S. Doplicher
R. Longo

Fondazione
C.I.M.E.

Springer

Editors and Authors

Alain Connes
Collège de France
11, place Marcelin Berthelot
75231 Paris Cedex 05, France

e-mail: connes@ihes.fr

Joachim Cuntz
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster, Germany

e-mail: cuntz@math.uni-muenster.de

Sergio Doplicher
Dipartimento di Matematica
Università di Roma "La Sapienza"
Piazzale A. Moro 5
00185 Roma, Italy

e-mail: doplicher@mat.uniroma1.it

Erik Guentner
Department of Mathematical Sciences
University of Hawaii, Manoa
2565 McCarthy Mall
Keller 401A, Honolulu
HI 96822, USA

e-mail: erik@math.hawaii.edu

Nigel Higson
Department of Mathematics
Pennsylvania State University
University Park, PA 16802
USA

e-mail: higson@psu.edu

Jerome Kaminker
Department of Mathematical Sciences
IUPUI, Indianapolis
IN 46202-3216, USA

e-mail: kaminker@math.iupui.edu

Roberto Longo
John E. Roberts
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica 1
00133 Roma, Italy

e-mail: longo@mat.uniroma2.it
roberts@mat.uniroma2.it

Cataloging-in-Publication Data applied for

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at http://dnb.ddb.de

Mathematics Subject Classification (2000): 58B34, 46L87, 81R60, 83C65

ISSN 0075-8434
ISBN 3-540-20357-5 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag is a part of Springer Science + Business Media

springeronline.com

Typesetting: Camera-ready TEX output by the authors

SPIN: 10967928 41/3142/du - 543210 - Printed on acid-free paper

Preface

If one had to synthesize the novelty of Physics of the XX century with a single magic word, one possibility would be "Noncommutativity".

Indeed the core assertion of Quantum Mechanics is the fact that observables ought to be described "by noncommuting operators"; if you wished more precision and said "by the selfadjoint elements in a C*-algebra $\mathcal{A}$, while states are expectation functionals on that algebra, i.e. positive linear forms of norm one on $\mathcal{A}$", you would have put down the full axioms for a theory which includes Classical Mechanics if $\mathcal{A}$ is commutative, Quantum Mechanics otherwise.

More precisely, Quantum Mechanics of systems with finitely many degrees of freedom would fit in the picture when the algebra is the collection of all compact operators on the separable, infinite-dimensional Hilbert space (so that all, possibly unbounded, selfadjoint operators on that Hilbert space appear as "generalized observables" affiliated with the enveloping von Neumann algebra); the distinction between different values of the number of degrees of freedom requires more details, as the assignment of a dense Banach *-algebra (the quotient, obtained by specifying the value of the Planck constant, of the L^1-algebra of the Heisenberg group).

Quantum Field Theory, as explained in Roberts' lectures in this volume, fits in that picture too: the key additional structure needed is the local structure of $\mathcal{A}$. This means that $\mathcal{A}$ has to be the inductive limit of subalgebras of local observables $\mathcal{A}(\mathcal{O})$, where $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ maps coherently regions in the spacetime manifold to subalgebras of $\mathcal{A}$.

As a consequence of the axioms, as more carefully expounded in this book, $\mathcal{A}$ is much more dramatically noncommutative than in Quantum Mechanics with finitely many degrees of freedom: $\mathcal{A}$ cannot be any longer essentially commutative (in other words, it cannot be an extension of the compacts by a commutative C*-algebra), and actually turns out to be a simple non type I C*-algebra.

In order to deal conveniently with the natural restriction to locally normal states, it is also most often natural to let each $\mathcal{A}(\mathcal{O})$ be a von Neumann algebra, so that

$\mathcal{A}$ is not norm separable: for the sake of both Quantum Statistical Mechanics with infinitely many degrees of freedom and of physically relevant classes of Quantum Field Theories - fulfilling the split property, cf. Roberts' lectures - $\mathcal{A}$ can actually be identified with a universal C*-algebra: the inductive limit of the algebras of all bounded operators on the tensor powers of a fixed infinite dimensional separable Hilbert space; different theories are distinguished by the time evolution and/or by the local structure, of which the inductive sequence of type I factors gives only a fuzzy picture. The actual local algebras of Quantum Field Theory, on the other side, can be proved in great generality to be isomorphic to the unique, approximately finite dimensional III_1 factor (except for the possible nontriviality of the centre).

Despite this highly noncommutative ambient, the key axiom of Quantum Field Theory of forces other than gravity, is a demand of commutativity: local subalgebras associated to causally separated regions should commute elementwise. This is the basic Locality Principle, expressing Einstein Causality.

This principle alone is "unreasonably effective" to determine a substantial part of the conceptual structure of Quantum Field Theory. This applies to Quantum Field Theory on Minkowski space but also on large classes of curved spacetimes, where the pseudo-Riemann structure describes a classical external gravitational field on which the influence of the quantum fields is neglected (cf Roberts' lectures). But the Locality Principle is bound to fail in a quantum theory of gravity.

Mentioning gravity brings in the other magic word one could have mentioned at the beginning: "Relativity".

Classical General Relativity is a miracle of human thought and a masterpiece of Nature; the accuracy of its predictions grows more and more spectacularly with years (binary pulsars are a famous example). But the formulation of a coherent and satisfactory Quantum Theory of all forces including Gravity still appears to many as one of the few most formidable problems for science of the XXI century.

In such a theory Einstein Causality is lost, and we do not yet know what really replaces it: for the relation "causally disjoint" is bound to lose meaning; more dramatically, spacetime itself has to look radically different at small scales. Here "small" means at scales governed by the Planck length, which is tremendously small but is there.

Indeed Classical General Relativity and Quantum Mechanics imply Spacetime Uncertainty Relations which are most naturally taken into account if spacetime itself is pictured as a Quantum Manifold: the commutative C*-algebra of continuous functions vanishing at infinity on Minkowski space has to be replaced by a noncommutative C*-algebra, in such a way that the spacetime uncertainty relations are implemented [DFR]. It might well turn out to be impossible to disentangle Quantum Fields and Spacetime from a common noncommutative texture.

Quantum Field Theory on Quantum Spacetime ought to be formulated as a Gauge theory on a noncommutative manifold; one might hope that the Gauge principle, at the basis of the point nature of interactions between fields on Minkowski space and hence of the Principle of Locality, might be rigid enough to replace locality in the world of quantum spaces.

Gauge Theories on noncommutative manifolds ought to appear as a chapter of Noncommutative Geometry [CR,C].

Thus Noncommutative Geometry may be seen as a main avenue from Physics of the XX century to Physics of the XXI century; but since it has been created by Alain Connes in the late 70s, as expounded in his lectures in this Volume, it grew to a central theme in Mathematics with a tremendous power of unifying disparate problems and of progressing in depth.

One could with good reasons argue that Noncommutative Topology started with the famous Gel'fand-Naimark Theorems: every commutative C*-algebra is the algebra of continuous functions vanishing at infinity on a locally compact space, every C*-algebra can be represented as an algebra of bounded operators on a Hilbert space; thus a noncommutative C*-algebra can be viewed as "the algebra of continuous functions vanishing at infinity" on a "quantum space".

But it was with the Theory of Brown, Douglas and Fillmore of Ext, with the development of the K-theory of C*-algebras, and their merging into Kasparov bivariant functor KK that Noncommutative Topology became a rich subject. Now this subject could hardly be separated from Noncommutative Geometry.

It suffices to mention a few fundamental landmarks: the discovery by Alain Connes of Cyclic Cohomology, crucial for the lift of De Rham Theory to the noncommutative domain, the Connes-Chern Character; the concept of spectral triple proved to be central and the natural road to the theory of noncommutative Riemannian manifolds.

Since he started to break this new ground, Connes discovered a paradigm which could not have been anticipated just on the basis of Gel'fand-Naimark theory: Noncommutative Geometry not only extends geometrical concepts beyond point spaces to "noncommutative manifolds", but also permits their application to singular spaces: such spaces are best viewed as noncommutative spaces, described by a noncommutative algebra, rather than as mere point spaces.

A famous class of examples of singular spaces are the spaces of leafs of foliations; such a space is best described by a noncommutative C*-algebra, which, when the foliation is defined as orbits in the manifold $\mathcal{M}$ by the action of a Lie group G and has graph $\mathcal{M} \times G$, coincides which the (reduced) cross product of the algebra of continuous functions on the manifold by that action. The Atiyah - Singer Index Theorem has powerful generalizations, which culminated in the extension of its local form to transversally elliptic pseudodifferential operators on the foliation, in terms of the cyclic cohomology of a Hopf algebra which describes the transverse geometry [CM].

There is a maze of examples of singular spaces which acquire this way nice and tractable structures [C]. But also discrete spaces often do: Bost and Connes associated to the distribution of prime numbers an intrinsic noncommutative dynamical system with phase transitions [BC]. Connes formulated a trace formula whose extension to singular spaces would prove Riemann hypothesis [Co]. The geometry of the two point set, viewed as "extradimensions" of Minkowski space, is the basis for the Connes and Lott theory of the standard model, providing an elegant motivation for the form of the action including the Higgs potential [C]; this line has been further

developed by Connes into a deep spectral action principle, formulated on Euclidean, compactified spacetime, which unifies the Standard model and the Einstein Hilbert action [C1].

Thus Noncommutative Geometry is surprisingly effective in providing the form of the expression for the action. But if one turns to the Quantum Theory it has a lot to say also on Renormalization. Connes and Moscovici discovered a Hopf algebra associated with the differentiable structure of a manifold, which provides a powerful organizing principle which was crucial to the Transverse Index Theorem; in the case of Minkowski space, it proved to be intimately related with Kreimer's Hopf algebra associated to Feynman graphs. Developing this connection, Connes and Kreimer could cast Renormalization Theory in a mathematically sound and elegant frame, as a Riemann - Hilbert problem [CK].

The relations of Noncommutative Geometry to the Algebraic Approach to Quantum Field Theory are still to be explored in depth. The first links appeared in supersymmetric Quantum Field Theory: the non polynomial character of the index map on some K groups associated to the local algebras in a free supersymmetric massive theory [C], and the relation to the Chern Character of the Jaffe Lesniewski Osterwalder cyclic cocycle associated to a super Gibbs functional [JLO,C].

More generally in the theory of superselection sectors it has long been conjectured that localized endomorphisms with finite statistics ought to be viewed as a highly noncommutative analog of Fredholm operators; the discovery of the relation between statistics and Jones index gave solid grounds to this view. While Jones index defines the analytical index of the endomorphism, a geometric dimension can also be introduced, where, in the case of a curved background, the spacetime geometry enters too, and an analog of the Index Theorem holds [Lo]. One can expect this is a fertile ground to be further explored.

Noncommutative spaces appeared also as the underlying manifold of a quantum group in the sense of Woronowicz; noncommutative geometry can be applied to those manifolds too. Most recent developments and discoveries can be found in the Lectures by Connes.

Noncommutative Geometry and Noncommutative Topology merge in the celebrated Baum - Connes conjecture on the K-Theory of the reduced C* algebra of any discrete group. While it has been realized in recent years that one cannot extend this conjecture to crossed products ("Baum - Connes with coefficients"), the original conjecture is still standing, a powerful propulsion of research in Index Theory, Discrete Groups, Noncommutative Topology. The lectures of Higson and Guentner expound that subject, with a general introduction to K-Theory of C*-algebras, E-theory, and Bott periodicity. Aspects of the Baum - Connes conjecture related to exactness are dealt with by Guentner and Kaminker.

K-Theory, KK-Theory and Connes - Higson E-Theory are unified in a general approach due to Cuntz and Cuntz - Quillen; a comprehensive introduction to these theories and to cyclic cohomology can be found in Cuntz's lectures.

Besides the fundamental reference [C] we point out to the reader other references related to this subject [GVF,L,M]. Since the theory of Operator Algebras is so intimately related to the subject of these Lecture Notes, we feel it appropriate to bring

to the reader's attention the newly completed spectacular treatise on von Neumann Algebras ("noncommutative measure theory") by Takesaki [T].

Of course this volume could not by itself cover the whole subject, but we believe it is a catching invitation to Noncommutative Geometry, in all of its aspects from Prime Numbers to Quantum Gravity, that we hope many readers, mathematicians and physicists, will find stimulating.

Sergio Doplicher and Roberto Longo

References

[BC] J.B. Bost & A. Connes, *Hecke Algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. **3**, 411-457 (1995).

[C] A. Connes, "Noncommutative Geometry", Acad. Press (1994).

[C1] A. Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Commun. Math. Phys. **182**, 155-176 (1996), and refs.

[Co] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. **5** (1999), no. 1, 29-106.

[CK] A. Connes, *Symetries Galoisiennes et renormalisation*, Seminaire Poincaré Octobre 2002, math.QA/0211199 and refs.

[CM] A. Connes & H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Commun. Math. Phys. **198**, 199-246 (1998), and refs.

[CR] A. Connes & M. Rieffel, *Yang Mills for noncommutative two tori*, in: "Operator Algebras and Mathematical Physics", Contemp. Math. **62**, 237-266 (1987).

[DFR] S. Doplicher, K. Fredenhagen & J.E. Roberts, *The quantum structure of spacetime at the Planck scale and quantum fields*, Commun. Math. Phys. **172**, 187-220 (1995).

[GVF] J.M. Gracia-Bondia, J.C. Varilly & H. Figueroa: "Elements of Noncommutative Geometry", Birkhaeuser (2000).

[JLO] A. Jaffe, A. Lesniewski & K. Osterwalder, *Quantum K-theory I. The Chern character*, Commun. Math. Phys. **118**, 1-14 (1988).

[L] G. Landi, "An Introduction to Noncommutative Spaces and their Geometries", Springer, LNP monographs 51 (1997).

[Lo] R. Longo, *Notes for a quantum index theorem*, Commun. Math. Phys. **222**, 45-96 (2001).

[M] J. Madore: "An Introduction to Noncommutative Differential Geometry and its Physical Applications", LMS Lecture Notes (1996).

[T] M. Takesaki, "Theory of Operator Algebras" vol. I, II, III, Springer Encyclopaedia of Mathematical Sciences 124 (2002), 125, 127 (2003).

CIME's activity is supported by:

Ministero dell'Università Ricerca Scientifica e Tecnologica, COFIN '99;
Ministero degli Affari Esteri - Direzione Generale per la Promozione e la Cooperazione - Ufficio V;
Consiglio Nazionale delle Ricerche;
E.U. under the Training and Mobility of Researchers Programme;
UNESCO-ROSTE, Venice Office.

Contents