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Heinz Hanßmann

Local and Semi-Local Bifurcations in Hamiltonian Dynamical Systems

Results and Examples

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to my parents

Preface

Life is in color,
But black and white is more realistic.
Samuel Fuller

The present notes are devoted to the study of bifurcations of invariant tori in Hamiltonian systems. Hamiltonian dynamical systems can be used to model frictionless mechanics, in particular celestial mechanics. We are concerned with the nearly integrable context, where Kolmogorov–Arnol'd–Moser (KAM) theory shows that most motions are quasi-periodic whence the (invariant) closure is a torus. An interesting aspect is that we may encounter torus bifurcations of high co-dimension in a single given Hamiltonian system. Historically, bifurcation theory has first been developed for dissipative dynamical systems, where bifurcations occur only under variation of external parameters.

Bifurcations of equilibria and periodic orbits

The structure of any dynamical system is organized by its invariant subsets, the equilibria, periodic orbits, invariant tori and the stable and unstable manifolds of all these. Invariant subsets form the framework of the dynamics, and one is interested in the properties that are persistent under small perturbations.

The most simple invariant subsets are equilibria, points that remain fixed so that no motion takes place at all. Equilibria are isolated in generic systems, be that within the class of Hamiltonian systems or within the class of all dynamical systems. In the latter case the dynamics is dissipative and an equilibrium may attract all motion that starts in a (sufficiently small) neighbourhood.

Such a dynamically stable equilibrium is also structurally stable in that a small perturbation of the dynamical system does not lead to qualitative changes. If we let the system depend on external parameters, then the equilibrium may lose its dynamical stability under parameter variation or cease to exist. A typical example is the $\mathbb{Z}_2$-symmetric pitchfork bifurcation where an

attracting equilibrium loses its stability and gives rise to a pair of two attracting equilibria. Other examples are the saddle-node and the Hopf bifurcation. Such bifurcations have been studied extensively in the literature, cf. [129, 173] and references therein.

The dynamics around equilibria in Hamiltonian systems can be more complicated since it is not generic for a Hamiltonian system to have only hyperbolic equilibria. This also influences possible bifurcations, cf. [61, 43]. For instance, in the Hamiltonian counterpart of the above pitchfork bifurcation it is an elliptic (rather than attracting) equilibrium that loses its stability and gives rise to a pair of two elliptic equilibria. In [254, 78] dynamically stable equilibria are studied for which the nearby dynamics nevertheless changes under variation of external parameters.

Periodic orbits form 1-parameter families in Hamiltonian systems, usually parametrised by the value of the energy. In fact, where continuation with respect ot the energy fails a bifurcation[1] takes place, while other bifurcations are triggered by certain resonances between the Floquet multipliers. For more details see [3, 38] and references therein, and also Chapter 3 of the present notes.

Bifurcation from periodic orbits to invariant tori

In (generic) dissipative systems periodic orbits are isolated and one needs again external parameters μ to encounter bifurcations. One of these is the periodic Hopf [154, 155] or Neĭmark–Sacker [252, 14] bifurcation. Under parameter variation a periodic orbit loses stability as a pair of Floquet multipliers passes at $\pm \exp(\mathrm{i}\nu T)$ through the unit circle, where T denotes the period. In the supercritical case the stability is transferred to an invariant 2-torus that bifurcates off from the periodic orbit, with two frequencies $\omega_1 \approx 1/T$ and $\omega_2 \approx 2\pi\nu$ coming from the internal and normal frequency of the periodic orbit. The subcritical case involves an unstable 2-torus with these frequencies that shrinks down to the periodic orbit and results in a "hard" loss of stability.

The frequency vector $\omega = (\omega_1, \omega_2)$ that in the above description is rather naively attached to the merging invariant tori exemplifies the problems brought by bifurcations to invariant tori. First of all we need non-resonance conditions $2\pi k/T + \ell\nu \neq 0$ for all $k \in \mathbb{Z}$ and $\ell \in \{1, 2, 3, 4\}$. Where these are violated one speaks of a strong resonance as the Floquet multipliers $\pm \exp(\mathrm{i}\nu T) \in \{\pm 1, -\frac{1}{2} \pm \frac{\mathrm{i}}{2}\sqrt{3}, \pm \mathrm{i}\}$ are ℓth order roots of unity, see [272, 173] for more details. While excluding these low order resonances does lead to an invariant 2-torus bifurcating off from the periodic orbit, the motion on that torus need not be quasi-periodic.

For irrational rotation number ω_1/ω_2 the motion is indeed quasi-periodic and fills the invariant torus densely. In case the quotient ω_1/ω_2 is rational (but

[1] For generic Hamiltonian systems this is a periodic centre-saddle bifurcation.

now with denominator $q \geq 5$) we expect phase locking with a finite number of periodic orbits with period $\approx qT$ and all other orbits on the torus heteroclinic between two of these. The invariance (and smoothness) of the torus is guaranteed by normal hyperbolicity, an important property of dissipative systems that does not have the same consequences in the Hamiltonian context.

In the present simple situation it suffices to require that the rotation number ω_1/ω_2 on the invariant torus has non-zero derivative with respect to the bifurcation parameter μ. A more transparent approach is to consider the rotation number as an additional external parameter and it is more convenient to work with both ω_1 and ω_2 as (independent and thus two) additional parameters. In (μ, ω)-space this yields the following description. The bifurcation occurs as μ passes through the bifurcation value $\mu = 0$ and the dynamics on the torus is quasi-periodic except where $\omega = (\omega_0 q, \omega_0 p)$ is a multiple $\omega_0 \in \mathbb{R}$ of an integer vector $(q, p) \in \mathbb{Z}^2$ and thus resonant.

Torus bifurcations in dissipative systems

Bifurcations involving invariant n-tori may similarly be described using external parameters $(\mu, \omega) \in \mathbb{R}^d \times \mathbb{R}^n$. An additional complication is that the flow on an n-torus may be chaotic for $n \geq 3$ and that the torus may be destroyed altogether in the absence of normal hyperbolicity. One therefore excludes resonances $k_1\omega_1 + \ldots + k_n\omega_n = 0$ by means of Diophantine conditions[2]

$$\bigwedge_{k \in \mathbb{Z}^n \setminus \{0\}} \quad | \, k_1\omega_1 \; + \; \ldots \; + \; k_n\omega_n \, | \; \geq \; \frac{\gamma}{|k|^\tau} \tag{0.1}$$

where $\gamma > 0$, $\tau > n - 1$ and $|k| = k_1 + \ldots + k_n$.

A first result along these lines concerns n-tori that bifurcate off from equilibria, cf. [23] and references therein. Here $d = n$ and the parameters μ are used to let n pairs $\mu_j \pm \mathrm{i}\omega_j$ pass through the origin $\mu = 0$ in μ-space. This yields quasi-periodic n-tori for ω in the nowhere dense but measure-theoretically large subset of $\mathbb{R}^n$ defined by (0.1), and also quasi-periodic m-tori where only $m < n$ pairs $\mu_j \pm \mathrm{i}\omega_j$ have crossed the imaginary axis.

Furthermore there are invariant tori of dimension $l > n$. In the simplest case $n = 2$ this has been proved in [32], establishing a quasi-periodic flow on the resulting 3-tori. The procedure in [24] does yield l-tori for general n, but no information on the flow on these tori.

Normal hyperbolicity yields invariant $(n+1)$-tori bifurcating off from a family of invariant n-tori in [68, 260, 119]. At the bifurcation the invariant n-tori momentarily lose hyperbolicity and the Diophantine conditions (0.1) are needed. As shown in [33, 34] one can similarly use Diophantine conditions

[2] The $\bigwedge$ at the beginning signifies that the inequalities that follow have to hold true for all non-zero integer vectors.

involving the normal frequency at the bifurcation to establish a quasi-periodic flow on the $(n+1)$-tori. The "gaps" left open where the frequency vector is too well approximated by a resonance are then filled by normal hyperbolicity. On this measure-theoretically small but open and dense collection of $(n+1)$-tori the flow remains unspecified. See also [55, 77] for more details.

Notably, these results require the bifurcating n-tori to be in Floquet form, with normal linearization independent of the position on the torus. The skew Hopf bifurcation where this condition is violated is a generic torus bifurcation that has no counterpart for periodic orbits. As shown in [282, 60, 62, 273] one has also in this case quasi-periodic $(n+1)$-tori bifurcating off from n-tori. The gaps left by the necessary Diophantine conditions are again filled by normal hyperbolicity, but to a lesser extent.

From the period doubling bifurcation [223, 173] of periodic orbits one inherits the frequency halving bifurcation of quasi-periodic tori. Under variation of the external parameter μ an invariant n-torus loses stability as a Floquet multiplier passes at -1 through the unit circle. In the supercritical case the stability is transferred to another n-torus that bifurcates off from the initial family of n-tori with the first[3] frequency divided by 2. The subcritical case involves an unstable n-torus with one frequency halved that meets the initial family and results in a "hard" loss of stability.

This situation is clarified in [34]. As μ passes through the bifurcation value $\mu = 0$ a frequency-halving bifurcation takes place for the Diophantine tori satisfying (0.1). By means of normal hyperbolicity the gaps around resonances $k_1\omega_1 + \ldots + k_n\omega_n = 0$ are filled by invariant tori on which the flow need not be conditionally periodic. This leaves small "bubbles" in the complement of Diophantine tori at and near the bifurcation value where normal hyperbolicity is too weak to enforce invariant tori. In [186, 187] this scenario has been obtained along a subordinate curve in the 2-parameter unfolding of a periodic orbit having simultaneously Floquet multipliers -1 and $\pm\exp(\mathrm{i}\nu T) \notin \{\pm 1, -\frac{1}{2} \pm \frac{\mathrm{i}}{2}\sqrt{3}, \pm\mathrm{i}\}$.

The quasi-periodic saddle-node bifurcation is studied in [65] where it appears subordinate to a periodic orbit undergoing a degenerate periodic Hopf bifurcation. The general theory is (again) given in [34], where it appears as the most difficult of the three quasi-periodic bifurcations inherited from generic bifurcations of periodic orbits. For an extension to the degenerate case see [284, 285].

Bifurcations in Hamiltonian systems

Compared to the above rich theory of torus bifurcations in dissipative dynamical systems, there are few results on conservative systems prior to [139] that I am aware of. In [41, 42, 32] invariant tori of dimension 2 and 3 are established in the universal 1-parameter unfolding of a volume-preserving vector

[3] Here a convenient choice of a basis on $\mathbb{T}^n$ is assumed.

field with an equilibrium having eigenvalues $0, \pm \mathrm{i}$ or $\pm \mathrm{i}\omega_1, \pm \mathrm{i}\omega_2$, respectively. In the Hamiltonian case the existence of invariant tori near an elliptic equilibrium is due to the excitation of normal modes and generalizes the Lyapunov centre theorem, see [55] and references therein.

This lack of a bifurcation theory for invariant tori in Hamiltonian systems is all the more surprising as no external parameters are necessary. Indeed, every angular variable on a torus has a conjugate action variable whence n-tori form n-parameter families. The present notes aim to fill this gap in the literature.

In the "integrable" case, when there are sufficiently many symmetries, the situation can be reduced to bifurcations of (relative) equilibria. For this reason we develop the latter theory in a systematic way. From the various families of equilibria one can easily reconstruct the bifurcation scenario of invariant tori in an integrable Hamiltonian system.

While integrable systems have received a lot of attention – not to the least because their dynamics *can* be completely understood – it is highly exceptional for a Hamiltonian system to be integrable. Still, one often takes an integrable system as starting point and studies Hamiltonian perturbations away from integrability. Also if explicitly given a non-integrable Hamiltonian system, one of the few methods available is to look for an integrable approximation, e.g. given by normalization, and to consider the former as a perturbation of the latter. By a dictum of Poincaré the problem of studying the effects of small Hamiltonian perturbations of an integrable system is the fundamental problem of dynamics.

KAM theory is a powerful instrument for the investigation of this problem. It states that most[4] of the quasi-periodic motions constituting the integrable dynamics survive the perturbation, provided that this perturbation is sufficiently (and this means *very*) small. In a more geometric language these motions correspond to invariant tori. Under Kolmogorov's non-degeneracy condition one may consider the (internal) frequencies as parameters, and the Diophantine conditions (0.1) bounding the latter away from resonances lead to the Cantor families of tori one is confronted with in the perturbed system.

In its first formulation KAM theory addressed the "maximal" tori, and only later generalizations were formulated and proven for families of invariant tori that derive from hyperbolic and/or elliptic equilibria. For an overview over this still active research area see [55]. The present notes further generalize these results to families of invariant tori that lose (or gain) hyperbolicity during a bifurcation. Such bifurcations are governed by the nonlinear terms of the vector field. In this way singularity theory both governs the bifurcation scenario and helps deciding how these nonlinear terms are dealt with during the KAM-iteration procedure. As a result, the various smooth families

[4] The relative measure of those parametrising internal frequencies for which the torus is destroyed vanishes as the size of the perturbation tends to zero.

of invariant tori of the integrable system get replaced by Cantor families of invariant tori organizing the perturbed dynamics.

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Utrecht, May 2006 *Heinz Hanßmann*

Contents

The sequential order in these notes is from equilibria to invariant tori, which means that the various types of bifurcations appear and re-appear in different chapters. Therefore the following overview on the main Hamiltonian bifurcations of co-dimension one might be helpful.

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