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Buildings of Spherical Type and Finite BN-Pairs

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Introduction

These notes are a slightly revised and extended version of mimeographed notes written on the occasion of a seminar on buildings and BN-pairs held at Oberwolfach in April 1968. Their main purpose is to present the solution of the following two problems:

(A) Determination of the buildings of rank $\geqslant 3$ and irreducible, spherical type, other than H_3 and H_4 ("of spherical type" means "with finite Weyl group", about the excluded types H, cf. the addenda on p. 274). Roughly speaking, those buildings all turn out to be associated to simple algebraic or classical groups (cf. 6.3, 6.13, 8.4.3, 8.22, 9.1, 10.2). An easy application provides the enumeration of all finite groups with BN-pairs of irreducible type and rank $\geqslant 3$, up to normal subgroups contained in B (cf. 11.7).

(B) Determination of all isomorphisms between buildings of rank $\geqslant 2$ and spherical type associated to algebraic or classical simple groups and, in particular, description of the full automorphism groups of such buildings (cf. 5.8, 5.9, 5.10, 6.6, 6.13, 8.6, 9.3, 10.4).

Except for the appendices, the notes are rather strictly oriented toward these goals. In particular, the general theory of buildings and BN-pairs is developed only inasmuch as it is relevant to the above problems (though, of course, we do not neglect to state and prove the auxiliary results in what seems to be their natural degree of generality, whenever this can be done without extra cost); further information can be found for instance in [12] (especially chap. IV and the exercises to it), [14], [55] and [95]. Also the motivation and applications of the theory are left out, except for a few words in the sequel of this introduction, where references will be given.

Concerning the prerequisites, we first mention that the part of the

notes which leads to the solution of problem (A), or at least to the assertion that the only buildings satisfying our requirements (rank $\geqslant 3$ etc.) are the known ones, is essentially self-contained: its reading only requires a basic knowledge of algebra. It might be observed that, because of this elementary character, some sections of the notes may be used as a simple introduction to some special chapter of algebra or geometry, for instance the theory of generalized quadratic forms (here called pseudo-quadratic forms: cf. 8.2), initiated in [96] and much used in differential topology, the geometry of conformal spaces (9.4) and of projective Cayley (or octonion) planes (9.11), etc. Less elementary is the §5 and everything that depends on it, namely the existence proofs for (A) and the solution of (B) in the case of exceptional groups; here, we use some basic facts of the theory of algebraic groups as can be found in [5], [8], [26], [96].

The origin of the notions of buildings and BN-pairs lies in an attempt to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups and, in particular, the exceptional groups ([77], [78], [79]). As is known, a complex projective space can be reconstructed from the group $SL_n(\underline{\underline{C}})$ as follows: the subspaces are the maximal parabolic subgroups (a subgroup is parabolic if it contains a maximal connected solvable subgroup) and two subspaces are incident in the geometrical sense if the intersection of the corresponding parabolic subgroups is also parabolic. This method for constructing a geometry from a group can be applied to any semi-simple complex Lie group and the main observation in the three papers mentioned above is that there exists a very simple algorithm by which, from the mere knowledge of the Dynkin diagrams of the groups, one can deduce basic properties of the associated geometries (for instance, the axioms of projective geometry in the case of SL_n) and relations between geometries associated to different groups. The fact that this result extends to real Lie groups and, more generally, to algebraic groups (of relative rank $\geqslant 2$ if one wants non-trivial statements) naturally lead to axiomatize the situation at two levels. At the geometric level, one wishes to associate to every Dynkin diagram a class of geometries in such a way that the algorithm in question above holds. The most obvious idea, followed in

[81] and [84] (cf. also [33]), consists in taking this algorithm (essentially, the proposition 3.12 of the present notes) as the main axiom. However, a simpler and more efficient definition, inspired by Chevalley's description of his groups [17] and, more specifically, by his proof of Bruhat's lemma was given in [89] and [95] and is adopted here for the introduction of the geometries in question, now called buildings (after Bourbaki [12]). At the group-theoretical level, the problem is to give conditions on a group G in order that, for a suitable notion of parabolic groups, the above construction provides a building. This leads to the axioms of BN-pairs given in [90] (cf. also [89], [93]). The notions of BN-pairs and buildings are not quite equivalent: to a BN-pair is naturally associated a building, practically by definition as we have just seen, but not all buildings arise in that way (though it turns out, *a posteriori*, that it is so in the irreducible, spherical rank $\geqslant 3$ case); on the other hand, different BN-pairs may give rise to the same building. The remarks we make hereafter on the possible generalization of problem (A) emphasize this difference between the two concepts. In recent times, the theories of buildings and BN-pairs have found applications beyond their initial purpose of interpreting geometrically the exceptional groups. The buildings - mostly those of spherical and of affine types - have proved a useful tool for the study of p-adic simple groups ([13], [14]), the cohomology of arithmetic groups ([6], [7], [34], [64]), the representation theory and harmonic analysis on finite and p-adic simple groups ([16], [65], [67], "special representation") and some questions in algebraic geometry ([47], [55]). As for the axioms of BN-pairs, they appear to be the natural frame to develop some chapters of the theory of simple groups "of Lie type", whether finite ([23], [24], [25], [43], [61]) or infinite and, in particular, p-adic ([13], [14], [40], [45], [50], [52], [93]). Prospects of further interesting developments are also opened by the construction, due to R.V. Moody and K.L. Teo [53] and to R. Marcuson [48], of BN-pairs, and hence buildings, whose Weyl groups are neither finite nor affine.

We now briefly comment on the meaning of the problems (A) and (B) and their solution.

Problem A is a kind of combinatorial analogue of the classification of algebraic simple groups of relative rank $\geqslant 3$. (In fact, that classification can easily be deduced from the solution of problem A, but the method given in [96] to obtain it, while less elementary, is more efficient; in particular, it is valid in principle for any rank and leads to a fairly explicit classification for all ranks $\geqslant 2$.) As for the applications, the main interest of problem A so far seems to lie in the classification of finite BN-pairs of irreducible type and rank $\geqslant 3$ (theorem 11.7). This result can be used as a short-cut to complete the proof of certain characterizations of finite simple groups of Lie type (cf. [36], [73], [106] and also [74] for general references): the method consists in showing that finite simple groups with preassigned group-theoretical properties (involving for instance centralizers of involutions or Sylow subgroups) have a BN-pair of rank $\geqslant 3$ and therefore must be of a known type by the classification theorem (though, as was pointed out by J. Thompson, once the BN-pair is constructed, one often already knows so much about the structure of the group that the reference to a general classification theorem is not really essential). In these notes, the theorem 11.7 is deduced as an easy consequence from the rather lengthy classification of buildings, but it certainly can be given a much shorter direct proof; actually, even the indirect proof given here is cut down to half of its length when one restricts oneself to considering only finite buildings (in that case, for instance, the § 9 becomes purposeless). The main ingredient of our solution of problem A is the rather technical theorem 4.1.2 which provides a kind of geometric substitute of the definition of groups with BN-pair as amalgamated sums (cf. 13.3 and 13.32): here, the "amalgamated objects" are buildings of rank 2. Once this theorem is proved, the classification of buildings of type A_n, D_n, E_6, E_7, E_8 becomes very easy, whereas the types C_n and F_4 still require much work. It would be possible to avoid the theorem 4.1.2 and give a more elementary case by case proof along the line followed in [78] and [90] (cf. also [82]) for A_n and E_6, but this is probably somewhat longer and gives less insight into the problem. Let us add a few words concerning the hypotheses: rank $\geqslant 3$ and, for the classification of BN-pairs, finiteness. The example of n° 11.14 shows that there can be no hope of classifying

infinite BN-pairs, even of such a harmless type as A_3. On the other hand, it is also out of question to determine all buildings of rank 2 : in the case of type A_2, that would mean classifying all projective planes and, even assuming finiteness, this seems to be beyond reach. To be mentioned here, however, is the remarkable theorem of W. Feit and G. Higman [29] according to which finite buildings of type $G_2^{(m)}$ (for the notation, cf. 2.19) do not exist for $m \neq 2, 3, 4, 6, 8$. As for the existing types, one has only very partial results and is often content with the mere construction of examples (see for instance [27], [57], [75], [83], [86] and, of course, the vast literature on projective planes). More hopeful is perhaps the problem of determining all finite BN-pairs of rank 2 ; an important step in that direction has been accomplished by P. Fong and G. Seitz [30] who have determined all finite BN-pairs (B,N) such that $B = (B \cap N).U$ where U is a nilpotent normal subgroup of B ("split" BN-pairs). Another promising field of investigation is provided by the "Moufang generalized polygons" (cf. p. 274) whose classification is perhaps within reach.

We have seen that the original purpose of the theory of buildings was the geometric interpretation of semi-simple groups and, in particular, of the exceptional groups. Clearly, this purpose is fully attained only if the given group can be identified to the full automorphism group of the associated geometry or at least to an easily characterizable subgroup of it. This is achieved by the solution of problem (B). Roughly speaking, the result is as follows: let G be either the group of rational points of an (adjoint) algebraic simple group over a field or a "projective" classical group (such as PGL, PGO, PGU, PGSp); then, if the building associated with G has rank $\geqslant 2$, the automorphism group of that building is the automorphism group of G, which is an extension of G by a group of automorphisms of the ground field (or the ground division ring) and possibly by some further obvious outer automorphisms ("diagram automorphisms"). This theorem can be viewed as an extension to buildings of the "fundamental theorem of projective geometry". A very nice application of it to the rigidity of locally symmetric spaces has been given by G. D. Mostow [54].

A word should now be said about the history of the results of these notes. We start again with problem (A). As was already mentioned above, a method was devised in [78] to derive various incidence properties of the geometry associated to a semi-simple group from the Dynkin diagram of that group. In the case of A_n, D_n, E_i, $(i = 6, 7, 8)$, that method makes it in principle possible to construct all buildings of the given types. For A_n, this is very easy and was practically done in [78] (cf. also [89]); the case of E_6 is worked out in [80]. In those papers, the results were not stated in the language we use here because the notion of buildings was not yet explicit. This notion was essentially introduced in [89] whereas the definition adopted here is given in [95]; in both papers, the relation with BN-pairs is indicated. Later on, BN-pairs became better known through their purely group-theoretical implications ([90] , [93]) so that, unaware of their geometric origin, several authors rediscovered the classification of BN-pairs of type A_n independently (cf. for instance [1] , [100]). At this point, it is appropriate to mention the work of C. W. Curtis [23] , who developed simultaneously and independently an axiomatic theory closely related to that of BN-pairs, but somewhat more complicated and less general, and who gave, in that framework, a complete classification of the groups of type A_n, C_n and E_i. The buildings of type C_n are almost equivalent to the "polar geometries" studied by F. D. Veldkamp [101] , who determined them all, except for the geometries of rank 3 whose planes are not Desarguesian (§ 9 of these notes) and for the geometries over division rings of characteristic 2 (it will be seen in § 8 that the treatment of that case requires the introduction of generalized quadratic forms). Using Veldkamp's work, G.L. Schellekens also gave an independent construction of all buildings of type D_n. In [94] were announced our result concerning the "presentation" of an arbitrary building of spherical type as an amalgamation of buildings of rank 2 (theorem 4.1.2) and, as an easy consequence (knowing all buildings of type A_3 and, by Veldkamp's theorem, all finite buildings of type C_3), the classification of all finite buildings and BN-pairs of irreducible type and rank $\geqslant 3$. Except for the §§ 9 and 10, the content of the present notes was exposed at the Oberwolfach meeting mentioned at the beginning of this introduction; the results concerning the types C_3 and F_4,

related to the work of H. Freudenthal on exceptional groups [32], were completed shortly afterwards and included in the original preprint version of the notes (1968).

As for problem (B), the main antecedents, besides the "fundamental theorem of projective geometry", are the theorems of W. L. Chow [19] and J. Dieudonné [28] (chap. III, §§ 3, 4) which essentially solve the problem for the classical groups of rank $\geqslant 3$ (the rank 2 case requires quite different methods: cf. § 8). To our knowledge, the only other previously investigated special cases are those of the Cayley planes ([31], [66], [76]) and their "split forms" ([46], [82], [102]).

As was already pointed out, a preprint version of these notes have been circulating since the fall of 1968. In the meantime, several papers have appeared where special cases of our results are proved; I did not try to keep track of them systematically, but those which came to my knowledge are mentioned in the bibliography. More than two thirds of the notes were already written in their final form about three years ago; though some proofs could be improved (cf. the addenda on p. 274) and the terminology is not quite up-to-date and not always consistent with that of [14]), no attempt was made to bring the desirable changes lest the publication process might diverge.

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