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The Souslin Problem

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C O N T E N T S

Chapter

CORRECTION

An Israeli mathematician called Uri Avraham has pointed out that, in Chapter IX, the claim straddling pages 105 and 106 is false as stated. To be precise, given an Aronszajn tree $\mathring{T}$, we define, for C a certain closed unbounded subset of ω_1, a (generic) order-embedding of $\mathring{T}|C$ into $\mathbb{Q}$, and claim that this order-embedding can be extended to all of $\mathring{T}$. As we have set things up this is false (although a slight modification to the definition of $\tilde{T}_{\alpha+2}$ on page 105 would in fact make our claim true). What we should have said is not that our particular embedding extends to all of $\mathring{T}$, but that it easily gives rise to such an embedding. In fact, this is perhaps more easily seen (via Theorem II.5) in terms of our original definition of special Aronszajn (Page 15):

By means of our embedding of $\mathring{T}|C$ into $\mathbb{Q}$, we may write $\mathring{T}|C = \bigcup_{m<\omega} A_m$, where each A_m is an (uncountable) antichain of $\mathring{T}$. Let $\langle a_\alpha^m | \alpha<\omega_1 \rangle$ be a one-one enumeration of A_m, each m, and let $\langle c_\alpha | \alpha<\omega_1 \rangle$ be the monotone enumeration of C. For each $\alpha< \omega_1$, and for each $x \in \mathring{T}_{c_\alpha}$, let $S^x = \{y \in \mathring{T}|c_{\alpha+1} \mid x <_{\mathring{T}} y\}$. Since each S^x is countable, let $\langle s_n(x) | n< \omega \rangle$ enumerate it. For each $n,m< \omega$, the set $B_{n,m} = \{s_n(a_\alpha^m) | \alpha< \omega_1\}$ is clearly an antichain of $\mathring{T}$. But $\mathring{T} = (\bigcup_{n<\omega} A_m) \cup (\bigcup_{n,m<\omega} B_{n,m})$, so $\mathring{T}$ is special. QED.

Keith J. Devlin
Bonn, 14.9.74

Introduction

In 1920, the awakening of interest in Set Theory was heralded by the apprarence of a new journal in mathematics - Fundamenta Mathematicae. It is fitting that in the very first volume appeared the statement of a problem which was to play a prominent role in the development of that subject. Indeed, as with the continuum problem, research into the simple question there raised by M. Souslin was to have far-reaching ramifications in several branches of set theory, notably in constructibility theory and in the theory of forcing, the very two subjects whose existence stemmed from the work on the continuum problem. As with the continuum problem, Souslin's problem concerned the real numbers. However, instead of asking "how many" reals there are, Souslin asked whether a certain set of conditions uniquely characterized the real numbers. More precisely, Souslin's problem was this. Suppose we are given a dense linearly ordered set X, complete (in the sense of, say, Dedekind cuts) and with no end-points, and that X has the property that there is no uncountable collection of pairwise disjoint open intervals of X. Must then X be (isomorphic to) the real line, $\mathbb{R}$? (Problème 3, M. Souslin [Su 1].) It turned out that this problem was undecidable on the basis of the usual (i.e. Zermelo - Fraenkel) axioms of set theory. It is the purpose of this short book to provide the proofs of this undecidability and of the undecidability of the problem even when the continuum hypothesis is assumed. The vast majority of these results are due solely to one person, Ronald Jensen. (In fact, Chapter VI is essentially the only place where results appear which are not his.) For the most part, these results appear here for the first time in print.

The exposition falls naturally into two parts. In the five first chapters, we discuss the non-provability of Souslin's hypothesis (i.e. the assumption that Souslin's problem has a positive answer) in ZFC $\pm$ CH. This part which also contains (Chapter II) a basic discussion of the problem, should be fairly easy to read. The prerequisites are a reasonable (e.g. 1st year graduate level) aquaintance with the method of forcing and a knowledge of constructibility theory. In chapter 1 we sketch briefly the material we require. It is natural to include in this part some discussion of the notion of a Souslin line (i.e. a "continuum" which does possess the properties stated in Souslin's problem,

but which is not isomorphic to $\mathbb{R}$), and Chapters IV and V provide just such a discussion. The second part - Chapters VI - X - is concerned entirely with the consistency of Souslin's hypothesis. Strictly speaking, the forcing theory outlined in Chapter I is all that is required for an understanding of the proofs, but some considerable aquaintance with this kind of material is really necessary. In particular, the proofs in this part tend to be extremely long and technical. We expect that chapters VIII - X will only be read in detail by the highly motivated (whatever that motivation may be !). These three chapters are certainly not the stuff of "first-year graduate courses" !

The book comes from two main sources. For the first part of the book, a set of lecture notes written by Jensen in Kiel in 1969. For the second part, a set of notes written by Devlin in Sidcup in 1972. These latter were based exclusively on a (much larger) set of notes written by Jensen some years earlier. For readers who have previously seen the original Jensen manuscript, let us say now that the proof of Con (ZFC + SH + CH) which we give here is essentially the same as the proof described there. The organisation of the proof is somewhat different, however, and the length is considerably reduced. Also (hopefully) we have found all the errors which the original manuscript contained.

We have tried to adopt the terminology and notation most common in current set-theoretical usage. We use 'iff' to denote 'if and only if' and use either QED or else ▌ to denote the end of a proof. Since this is not really a textbook (rather an account of Jensen's work) we have not included any exercises. The reader who completes the entire book will not need any further exercise !

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