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Dedicated to

GEORG KREISEL

who has contributed so much to the
subject of this volume

Preface

The present volume found its origin in a course on functional and realizability interpretations on intuitionistic formal systems, presented at the Rijksuniversiteit Utrecht (Netherlands) in the spring of 1970, and a course on the metamathematics of intuitionistic formal systems at the University of Amsterdam in 1971 - 1972. The literature on the subject was widely scattered, the connection between certain rules was often not made explicit in the literature, and some obvious questions were not answered there.

Therefore I thought it would be useful to give a coherent presentation of the principal methods for metamathematical investigation of intuitionistic formal systems and the results obtained by these methods, connecting results in the literature, filling gaps and adding some new material. A first attempt (for realizability and functional interpretations) was made in Troelstra 1971, which, however, because of a rather terse style, was not readily assimilated by readers new to the field. (It still provides a useful survey of the applications to first-order systems however.) Therefore a more elaborate presentation, including other techniques of metamathematical research, seemed to be called for.

Having learnt of the unpublished Ph.D. work of C.Smorynski on applications of Kripke-models to intuitionistic arithmetic, and of Dr Zucker's thesis on the intuitionistic theory of higher-order generalized inductive definitions, subjects which both fitted very well into the scope of the planned volume, I asked them to contribute a chapter each; their contributions appear as chapters V, and VI respectively. The models for intuitionistic arithmetic of finite type, functional and realizability interpretations, and normalization for natural deduction systems, and also the general editing of the volume I undertook myself.

Finally, W.A. Howard contributed an Appendix supplementing discussions in § 2.7 and § 3.5.

The organization of the volume is primarily method-centered, i.e. the material presented is grouped mostly around methods and techniques, and not arranged according to the results obtained. Hence some results, obtainable by different methods, appear at various places in the book. This will enable the reader to compare the relative merits of the various methods.

As regards intuitionistic arithmetic and closely related systems, the treatment is almost wholly self-contained; some experience with classical

metamathematics, and the elements of intuitionism, such as may be gleaned from Kleene's Introduction to metamathematics and Heyting's book on Intuitionism suffices. The parts dealing with arithmetic can therefore be used in a course for graduate students or a seminar.

The sections dealing with analysis are not self-contained, and serve more or less as a running commentary on the literature, connecting and comparing various approaches and adding new results besides. This part was thought of primarily as a help to the beginning researcher, to help him to find his way in the subject. For use in a seminar, these sections should usually be supplemented by the reading of other papers.

In keeping with this set-up, the listing of applications for intuitionistic arithmetic and closely related systems is rather extensive, but in the case of analysis we have often restricted ourselves to some typical examples ; further applications can easily be made by the reader himself once he has understood the method, and its applications to arithmetic.

No special attention has been given to intuitionistic propositional logic and predicate logic, because as formal systems they exhibit many properties which do not generalize to arithmetic and analysis, and therefore would require a separate treatment.

Speedy publication was thought more useful than final polish, so as not to make the material outdated at the moment of its appearance. Hence also the choice for publication in the "Lecture Notes in Mathematics". Even while refraining from a completely self-contained treatment of all parts, it was not possible to take all relevant work into account, not even on arithmetic ; for example, N. Goodman's work on the theory of constructions was left out altogether, since it would not easily be fitted into the framework of the other developments and so would consume too much space.

We have no doubt that there are still many imperfections in this presentation ; it hardly needs saying that the authors will be grateful for errors, misprints, additions to the bibliography being brought to their attention.

The contents of the present volume are primarily technical in character ; but it is to be hoped that the material will not inspire a thought- and mind-less multiplication of metamathematical results, without a thought spent on their possible significance for an analysis of intuitionistic basic notions and for foundations of mathematics in general. On the other hand, the "philosophical interest" of the subject is not promoted by uncritical analysis. (A single example : the interest of the well known disjunction property $\vdash A \vee B \Rightarrow \vdash A$ or $\vdash B$, and the explicit definability for existential statements are frequently overrated, especially as a criterion for the

"constructive character" of the system considered. See e.g. the discussion in Troelstra A.) As regards potential "philosophical interest", it seems to me to be more promising (but also more difficult) to look for new results for well-known systems (possibly different in kind from the results discussed in this volume), instead of trying to extend known results to stronger and stronger systems. Of course, to be potentially interesting, the new results should also have a clear intuitive meaning in terms of the intended interpretation of the systems considered.

Directions for use. In order to help the reader find his way, there is an analytical table of contents at the beginning, a bibliography, and lists of notions and notations at the end. Reference to the bibliography are self-explanatory. § 3.5 refers (except in the appendix) to chapter III, § 5, etc.

The parts on arithmetic and closely related systems are more or less self-contained. As such we mention especially: Chapter I, §§ 1 - 8, §§ 10, 11; chapter II, §§ 1 - 4 (2.4.18 excepted), § 5, § 7 (except where results of § 6 are used); chapter III, § 1 (3.1.1 - 18), § 2 (3.2.1 - 28; 3.2.33), § 4 (3.4.1 - 14; 3.4.29), § 5 (3.5.1 - 11; 3.5.16 (i), (iii)); § 6 (3.5.1-3.6.16), § 7 (3.7.1 - 8), § 8 (except 3.8.7), § 9; chapter IV, §§ 1 - 4; chapter V, §§ 1 - 6.

Chapter I contains all generalities, and should usually be consulted when needed only.

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