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M. Bunge · J. Funk

Singular Coverings of Toposes

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Dedicated to F. W. Lawvere

Preface

This book gives an introduction to a theory of complete spreads, which basically uses the same strategy employed by R.H. Fox [Fox57] for dealing with branched coverings, and which we carry out in close connection with (and parallel to) a theory of distributions in the sense of F. W. Lawvere [Law66, Law83, Law92].

Rather than elucidating the concepts of toposes, distributions, and complete spreads in this preface, we give a preliminary taste of these concepts by including certain quotations which authoritatively describe the original settings and motivations behind them, and let the reader explore these concepts and their interplay in the often new guises in which we present them in this book, itself based on our own work on these topics during the past ten years.

Toposes

"The original notion of a *topos*, as a 'generalized space' suitable for supporting the exotic cohomology theories required in algebraic geometry, sprang from the fertile brain of Alexandre Grothendieck in the early 1960's, and was developed in his Séminaire de Géométrie Algébrique du Bois-Marie particularly during the academic year 1963-64. The duplicated notes of that seminar circulated widely among algebraic geometers and category-theorists over the next decade, until Springer-Verlag did the world a service by publishing a revised and expanded edition in three volumes of *Lecture Notes in Mathematics* in 1972 [AGV72b]. But by then the subject had already been 'reborn' in its second incarnation, as an elementary theory having links with higher-order intuitionistic logic, through the collaboration of Bill Lawvere and Myles Tierney during 1969-70 (which, in turn, built upon Lawvere's work on providing a categorical foundation for mathematics, which had been developing since the early 1960's)."

P. T. Johnstone, *Sketches of an Elephant. A Topos Theory Compendium.* Volumes 1 and 2, Oxford University Press, 2002.

Distributions

"Following up on a 1966 Oberwolfach talk where I had proposed a theory of distributions (not only in but) on presheaf toposes, in 1983 at Aarhus I posed several questions concerning distributions on $\mathscr{S}$-toposes [where $\mathscr{S}$ is an elementary topos, thought of to be *Set*, the category of sets and functions]. The base for the definition and questions is a pair of analogies with known theories (commutative algebra and measure theory) for variable quantities, coupled with the fact that there are many important examples of variable $\mathscr{S}$-'quantities' where the domains of variation are $\mathscr{S}$-toposes. The intensively variable quantities are taken to be the sheaves on the topos, i.e., simply the objects in the category. Of course, the term 'topos' means 'place' or 'situation', but Grothendieck treats the general situation by dealing instead with the category of *Set*-valued quantities which vary continuously over it, as an affine k-scheme is described by dealing with the k-algebra of functions on it. (...) Then we follow the lead of analysis and define a *distribution* or extensively variable quantity on an $\mathscr{S}$-topos to be a continuous linear functional, or generalized point, i.e., a functor to $\mathscr{S}$ which preserves $\mathscr{S}$-colimits, but not necessarily the finite limits."

F. W. Lawvere, Comments on the Development of Topos Theory, in: Jean-Paul Pier (editor), *Development of Mathematics 1950-2000*, Birkhäuser Verlag, Basel Boston Berlin, 2000, 715-734.

Complete Spreads

"The principal object of this note is to formulate as a topological concept the idea of a 'branched covering space'. This topological concept encompasses the combinatorial concept used by Heegard, Tietze, Alexander, Reidemeister and Seifert. This has as a consequence that the knot-invariants defined by Seifert (the linking invariants of the cyclic coverings) are invariants of the topological type of the knot (i.e., unaltered by an orientation-preserving autohomeomorphism of 3-space). Without the developments of this note I am unable to see any simple proof that these invariants are invariants of anything more than the combinatorial type of the knot. It appears that the best way to look at a branched covering is as a *completion* of an unbranched covering. This completion process appears in its simplest form if it is applied to a somewhat wider class of objects. It is for this reason that I introduce the concept of a *spread* (a concept that encompasses, in particular, the 'branched and folded coverings' of Tucker.)"

R. H. Fox, Covering Spaces with Singularities, in R. H. Fox et al. (editors), *Algebraic Geometry and Topology; A Symposium in Honor of S. Lefschetz*, Princeton University Press, 1957, 243-257.

Acknowledgments

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Montréal and Bridgetown, February 2006

Marta Bunge
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Introduction

Our main objective in this book is to develop a fairly self-contained theory of certain singular coverings of toposes which we name complete spreads. For instance, locally constant coverings are complete spreads, and are essential in describing those complete spreads which are branched coverings. Our theory extends the complete spreads in topology due to R. H. Fox (1957) but, unlike the classical theory, it emphasizes an unexpected connection with topos distributions in the sense of F. W. Lawvere (1983): topos distributions and complete spread geometric morphisms with a locally connected domain are opposite sides of the same coin.

We think of complete spreads as the geometry of distributions. The two notions complete spread and distribution come together and reinforce each other. Our constructions, though often motivated by classical theories, are sometimes quite different from them. We study special classes of distributions and of complete spreads, inspired respectively by functional analysis and topology. Among the former are the probability distributions; the branched coverings are singled out amongst the latter.

An étale geometric morphism, or local homeomorphism, is familiar to every topos-theorist. A complete spread is a kind of geometric morphism that is dual to an étale geometric morphism. Informally, if an étale geometric morphism is a generalized open part of a topos frame, then a complete spread is like a generalized closed part. Here we think of "closed" as orthogonal to a naturally prior notion of dense, as in a factorization system, rather than as the formal complement of open.

Just as sheaves and local homeomorphisms form equivalent categories, so do distributions (cosheaves) and complete spread geometric morphisms. In fact, the two equivalences may be established by the same basic construction, but they differ wildly in several respects. For instance, local homeomorphisms are exponentiable, but complete spreads need not be. Complete spreads are as basic and fundamental as local homeomorphisms. We hope that the reader will gain an appreciation of this, and of how we have come to this view.

Throughout we work in, or over, an elementary topos $\mathscr{S}$, of which we usually make no special assumptions. Thus, our methods and results are necessarily constructive and relative to $\mathscr{S}$. This perspective is important for other reasons. For instance, it helps to clarify and understand separately the completeness condition.

The symmetric topos classifies distributions and is part of a Kock-Zöberlein-monad (KZ-monad, for short) M on the 2-category $\mathbf{Top}_{\mathscr{S}}$ of toposes bounded over $\mathscr{S}$, geometric morphisms over $\mathscr{S}$, and 2-cells between them. We develop an axiomatic theory of certain KZ-monads (completion, closed, linear) on a 2-category $\mathscr{K}$, of which M is the motivating example. In particular, this gives a theory of complete spreads as an instance of the theory of discrete fibrations relative to a particular one such KZ-monad, namely the symmetric KZ-monad on $\mathbf{Top}_{\mathscr{S}}$.

We specialize and exemplify our theory in several different contexts: algebraic, logical, topological. In particular, we begin to develop a theory of branched coverings as certain kinds of complete spreads.

The present work stems from (but is not limited to) a number of papers written by the authors, individually, and in collaboration with each other or with others, since 1995. In the process of almost totally reorganizing and revising the existing material in a coherent manner, we have included numerous new concepts and results, new proofs of old results, and new examples from various areas of mathematics.

Although primarily aimed at topos theorists, this book may also be used as a textbook for an advanced one-year course introducing topos theory with an emphasis on geometric applications.

The book is divided into three interrelated, yet distinct parts, both in content as well as in background and approach. They are meant to be read in the order in which they are presented. However, an alternative way of discussing the material in two one-semester courses, with the second as a follow-up of the first, would be (roughly) to devote the first course to chapters 1, 2, 7, and 9, relegating chapters 3, 4, 5, 6, and 8 to a second course. Another possibility would be to cover chapters 1, 2, 3, 4, 5, ancd 6 in the first part, leaving chapters 7, 8, and 9 for the second part of the year, in view of the several open problems stated in the latter chapters.

In *Part 1* we introduce distributions and complete spread geometric morphisms, motivating them by means of examples. We then establish the main fact in this book, namely that there exists an adjoint equivalence between distributions on a topos $\mathscr{E}$ and complete spreads maps over $\mathscr{E}$ with locally connected domain. We emphasize that our investigations of complete spread geometric morphisms benefit from this far-reaching equivalence. Thus, the distribution concept from functional analysis and a concept from topology, namely complete spread, come together in the topos realm to form a coherent theory. The analogies multiply. Just as sheaves and cosheaves are dual notions, so are local homeomorphisms and complete spreads, functions and distributions, and discrete fibrations and discrete opfibrations. The rest of the

book is devoted to an exploration of these aspects and to giving interesting examples and applications in various fields of mathematics.

Local connectedness plays a central role. Barr and Paré's (1980) investigation into local connectedness points to the relevant notion of complemented subobject: what they term a definable subobject (more generally, definable morphism). It is easy to see that a locally connected geometric morphism is subopen, and that definable morphisms compose in its domain topos. Such a geometric morphism is therefore what we name a definable dominance. These conditions form part of the Barr-Paré characterization of local connectedness.

The 'comprehensive factorization' is one of our basic tools. It says that every geometric morphism with locally connected domain admits a unique factorization into a pure geometric morphism followed by a complete spread.

We also deal separately with the notions of spread and completeness, and prove that a geometric morphism is a complete spread if and only if it is both a spread and complete.

In *Part 2* we develop an abstract (axiomatic) theory of complete spreads. Distributions on a topos $\mathscr{E}$ have a classifier: the symmetric topos. The study of the symmetric topos as the functor part of a monad is an instance of what we call a completion Kock-Zöberlein monad, or completion KZ-monad, for short. From certain key axioms (completion, closed, linear), which are all satisfied by the basic model, namely, the symmetric monad M, we are able to deduce, by using only 2-category theory, all of the main theorems of the subject, such as the comprehensive factorization and in turn, the equivalence between distributions on a topos $\mathscr{E}$ (or points of the topos $\mathrm{M}(\mathscr{E})$), and complete spreads over $\mathscr{E}$ with locally connected domain (or discrete M-fibrations over $\mathscr{E}$), where M is the symmetric monad. For instance, we show with these methods that complete spreads with locally connected domain are stable under bipullback along an $\mathscr{S}$-essential geometric morphism. It is not clear to us how to prove this directly from the original definition of complete spread.

There is a natural notion of density of a distribution that may be cast within the theory of closed completion KZ-monads. In connection with this, we make a special assumption on an object B of the 2-category $\mathscr{K}$ on which a given completion KZ-monad is given, namely, the existence of a Gleason core of B, or equivalently, the existence of the terminal distribution on B, meaning the terminal point of $\mathrm{M}(B)$. The 'Gleason core axiom' holds for all Grothendieck toposes, but a general criterion for when does it hold in more general situations is not available.

The pullback definition of a complete spread is used to establish an intrinsic characterization of completeness saying that a geometric morphism's pure factor is a surjection iff a certain cover-refinement property holds. The bicomma construction of a complete spread, on its part, shows that this same property (the pure factor is a surjection) holds iff every cogerm comes from a point, or that 'cogerms converge.' Thus, cogerms converge iff the cover-refinement property holds. This statement may be interpreted as a sort of 'Heine-Borel theorem'.

Lawvere has posed the question of finding a suitable 'single universe' in which cosheaves and sheaves exist and interact . For instance, the Kleisli category of the symmetric monad M is a natural candidate for such a single universe. Another approach to single universes uses a glueing construction along the density functor from distributions (respectively complete spreads, cosheaves, closed parts) to functions (respectively local homeomorphisms, sheaves, open parts), when such exists.

In *Part 3* we deal with further aspects of distributions and complete spreads, such as localic, algebraic, lattice-theoretic, and topological.

We begin with the lattice-theoretic aspects. A distribution on an object $\mathscr{E}$ of $\mathbf{Top}_{\mathscr{S}}$ is a pair of adjoint functors $\mu \dashv \mu_*$ in the sense of $\mathscr{S}$-indexed categories. We usually identify the notion of a distribution with just the $\mathscr{S}$-cocontinuous functor $\mu : \mathscr{E} \longrightarrow \mathscr{S}$ since, by the indexed version of the (special) adjoint functor theorem, such a μ must have an $\mathscr{S}$-indexed right adjoint μ_*. We show that the distribution μ is completely determined by the 'distribution algebra' $H = \mu_*(\Omega_{\mathscr{S}})$. Distribution algebras are part of a duality similar to the classical Stone duality. Indeed, the duality between *Set* and complete atomic Boolean algebras is a special case of the distribution algebra duality. We prove for instance that the opposite of the category distributions on a Grothendieck topos is monadic over the given topos.

Pure geometric morphisms are also the first factor of our relative version of Johnstone's (1982) pure, entire factorization theorem, in which we introduce relative versions of Stone locale and entire geometric morphism. A notion of relative Boolean algebra due to Jibladze and investigated further by Kock and Reyes (1999) is relevant in these investigations. Since a distribution algebra H in $\mathscr{E}$ over $\mathscr{S}$ is in particular an $\Omega_{\mathscr{S}}$-Boolean algebra, we may also consider the pure, entire factorization (relative to a base topos $\mathscr{S}$), and compare it with the comprehensive factorization for a geometric morphism with a locally connected domain.

In what we view as localic and algebraic aspects of distributions, we begin with an analysis of two important examples of completion KZ-monads that are related to the symmetric monad M. One is the lower power locale P_L, of interest in theoretical computer science, and the other (also of interest in theoretical computer science, but not just therein) is the lower bagdomain topos B_L. Not surprisingly, we obtain the decomposition $\mathrm{M} = \mathrm{B}_L \cdot \mathrm{T}$, where T classifies probability distributions. The KZ-monad T is thus the difference between distributions and bags of points. In this connection, a notion of discreteness that we call 'discrete complete spread structure' emerges as yet another single universe for local homeomorphisms and complete spreads. We discuss and analyse other algebraic aspects of distributions, such as coschemes. We make a special analysis of distributions on the so-called Jonsson-Tarski topos.

In the final chapter on topological aspects of distributions and complete spreads, we deal with branched coverings and, in particular, with locally constant coverings. We do not attempt to formally deal with folded coverings in this context, although they too ought to be instances of spreads. Our definition

of a branched covering is inspired by the classical construction of a branched covering of a space with a given set of singularities as the spread completion of a universal (locally trivial) covering of the non-singular part of the space. Implicit in such a definition is the existence and uniqueness of the completion of a spread. We characterize branched coverings (with a locally connected domain) in an axiomatic way that relies only on the notions of complete spread, locally trivial covering, pure subobject, and on a newly isolated notion of purely skeletal geometric morphism. In effect, in this book we make the latter our formal definition of a branched covering and then prove its equivalence with the classically-inspired notion.

We also explain the difference between the usual covering spaces (locally constant, or locally trivial) and the unramified coverings which, in our context, form precisely the intersection of the two main classes of geometric morphisms that we have been considering: local homeomorphisms and complete spreads. We find something common to the usual coverings and the unramified ones: they both satisfy a general van Kampen theorem with respect to the same class of geometric morphisms of effective descent, namely, locally connected surjections. We discuss the fundamental groupoid as well as the branched fundamental groupoid, but we leave aside questions pertaining to path lifting and homotopy as this would take us too far from the main subject of the book.

Throughout we emphasize open problems whenever it seems appropriate to do so. Several routine proofs are left as exercises, but also as ‘exercises’ the reader will find open questions for possible future work. We leave it to the reader to judge the difficulty involved in each problem posed, partly because we do not always have a complete appraisal of it ourselves, but also because difficulty is a subjective matter, not inherent to the questions themselves.