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120

Decidable Theories I

Edited by Gert H. Müller, Heidelberg

Dirk Siefkes

Mathematisches Institut der Universität, Heidelberg

Büchi's Monadic Second Order Successor Arithmetic

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To Marie Luise

Editor's Preface

For several years I conducted a Seminar on decision procedures of various mathematical theories at the Mathematical Institute of the University of Heidelberg, some of the time jointly with Professor O. Herrmann. It is planned that I should continue this seminar with Dr.D.Siefkes. The results will be published in a subseries of "Lecture Notes in Mathematics", to be called "Decidable Theories". The following aspects were, and will be, decisive both for the seminar and the Lecture Notes arising out of it.

(i) The decidable theory in question and the decision procedure are set up in purely syntactical terms, hence not referring, for instance, to the true sentences of a preferred model. The transformation steps which in the simplest case lead to an equivalence with the propositional constants T (true) or F (false) are proved within the theory in question, - of course, using syntactical means.

(ii) A presentation of a decision procedure in the sense of (i), provides a set of operational devices which, if applied to a sentence, leads to its decision. From such devices certain syntactical parameters of a sentence can be exhibited, and in these terms an estimation of the complexity of the decision procedure can be made. - Obviously, most presentations of decision procedures in the literature are given so that the decidability is proved in the simplest mathematical way. Hence it is not surprising that a presentation aiming at practical applications results in some remarkable simplifications of the decision procedure.

(iii) A decision procedure usually is: 1) the reduction of an arbitrary sentence to a normal form (using the axioms of the theory) and 2) the reduction of a given sentence in normal form to T or F, as usual. - Relative to a preferred model, e.g. interesting from the point of view of applications, the normal form indicates the type of question which can be answered by the use of the decision procedure. There may be different normal forms for one theory, to which a suitable reduction can be applied; hence different types of questions may be decidable. It is of interest to find questions in science or technology which are reducible to such types.

The aim of the Lecture Notes on Decidable Theories is a systematic and almost self-contained presentation of the known decision procedures, taking into account (i)-(iii). It should in principle be possible to program a decision procedure from such a presentation. Substantial simplifications may arise if certain syntactical parameters are numerically given in advance. - It is the intention of the Series to bridge the gap between applications and the more or less abstract results on decidability.

Heidelberg, January 6, 1970 Gert H. Müller

CONTENTS

Introduction

The subject of these lecture notes is Büchi's decision procedure (see [3]) for his Sequential Calculus SC, i.e. for the monadic second order fragment of arithmetic which involves only the successor function, but neither addition nor multiplication. We set up a decision procedure for SC in purely syntactical terms, using none but the three Peano axioms for successor. As a result we prove the Post-completeness (syntactical completeness) of this system, and obtain several measures towards practical effectivity of the decision procedure (chap.I). In chap.II we investigate definability in SC. We introduce three extensions of the concept "ultimately periodic" from sets of natural numbers to relations, and prove that one of them is appropriate for SC, namely that just the "fanlike ultimately periodic" relations are definable in SC. This result provides especially a characterization of the functions definable in SC, and yields effective quantifier elimination for SC-formulae without free predicate variables. (Chap.II.2.)

In a monadic second order theory one has at hand quantifiable one-place predicate variables, besides the means of first order logic. During the last ten years monadic second order theories have turned out to be a very powerful tool in establishing decidability. The bound predicate variables allow to express a lot of combinatorial facts within the theory; e.g. all known decision procedures for monadic second order theories use finite automata on finite or infinite tapes or trees, by translating automata theory into the investigated theory. The engagement between automata and monadic second order started in the mid-fifties from several papers of Church who uses quantifier-free fragments of monadic second order arithmetic as "condition languages" for automata theory (see [8]-[10]). The marriage was performed by Büchi; in his papers [2]-[5] (see also [6] and [7]) for the first time the decidability of several monadic second order theories is shown, and for the first time finite automata on infinite tapes are used for such proofs. It is easily seen, and is done in the author's Diplom paper, that all known decision procedures (and also a lot of new ones) for fragments of arithmetic can be derived from Büchi's decision procedure [3] for SC. How fruitful the mariage was, became even more evident in the papers of Elgot-Rabin [12] and Rabin [28] and [29]. In [28], Rabin solves a lot of famous decision problems in the affirmative by proving the decidability of the monadic second order theory of two (or of more) successors; his main tool is the theory of automata on infinite trees. - Thus it seems worthwhile to investigate decidable monadic second order theories. Especially, SC calls for a closer inspection: on the one hand, most of the consepts useful in monadic second order theories are introduced already in [3]; on the other hand, Büchi's decision procedure is published only as a very short congress talk paper which leaves a lot of work to the reader.

A decision procedure for a theory is a mechanical procedure which, applied to any sentence of the theory, tells after finitely many steps whether the sentence is true or not. In most cases, a decision procedure is given semantically, i.e. the true sentences of the theory are predescribed somehow, and then model-theoretical methods are used to show how to transform any sentence of the theory step by step into an "evidently" true or false one. To get a syntactical version of the decision procedure one has to write down within the language of the theory all transformations of formulae which appear in these steps; especially one has to translate all means from outside into principles which are expressible within the used language. The thus collected transformations constitute an axiom system from which any true sentence of the theory is derivable. Therefore the best characterization of the theory would be a simple part of this axiom system from which the remaining transformations are derivable. - Such a "syntactization" indirectly gives a further account of the strongness or weakness of the theory. Namely it shows what theorems are derivable or not derivable in the theory. In general, it leads to certain normal forms for formulae, and thus marks off the "range of questions" of the theory; i.e. one sees more clearly what sort of problems can be formulated in the theory.

In chap.I, this program of syntactization is carried through for the Sequential Calculus SC. Büchi sets up semantically both his system and the decision procedure. As means from outside he uses results from the theory of finite automata, and the famous combinatorial theorem A of Ramsey [31]. (It should perhaps be pointed out that by syntactization one does not get rid of these means from outside. Syntactization just makes the proofs "elementary", i.e. expressible in the language, and thereby shows the degree of complexitiy of the theory. But the elementarized proofs mostly are very cumbersome, and not understandable but from their intuitive (= outside) formulation.)

The main points of chap. I are:

(i) In §§1-3 a syntactical decision procedure for SC is presented - for the first time, as far as we know. Clearly the procedure follows Büchi's semantical procedure, in the manner described above.

(ii) It is shown that on the background of monadic second order logic from the three Peano axioms for the successor function, recursion theory as far as it is expressible can be built up in the system. This part of recursion theory suffices to derive the Ramsey theorem, suggested by Büchi as an axiom, and to replace automata theory. At one place, recursion formulae, which work like finite automata in the system, allow to avoid at all the most involved contribution from automata theory. Thus it is shown that this very simple axiom system is complete. Derivation of recursion theory and of the Ramsey theorem in SC are to be found already in [36] - however, in a more complicated version than here.

(iii) In §4 the decision procedure is inquired with regard to feasibility. - A step-by-step-description of the whole procedure is given, to encourage people interested in application but not in proofs, to try to program the procedure on a computer. Estimations of the growing of the length of formulae under the transformations of the decision procedure indicate the most awful parts of the decision procedure (awful with regard to practical application), and give reason to quite a lot of improvements. E.g. we replace at a central place of the decision procedure certain formulae used by Büchi by simpler ones, and lower thereby a growing rate of 2^{2^a} to 2a. In 5.c, it is just the most intrinsic part of the completeness proof, which suggests how to replace another part of the decision procedure by a simple combinatorial consideration. These improvements together make it more likely that practical application of the decision procedure is possible.

(iv) Most effort is made to make clear the intuitive background of the decision procedure. Tuples of predicates are regarded as sequences of tuples of truth values, and SC-formulae containing free predicate variables are regarded as conditions on such sequences, called "threads". The research on nerve nets once has led to the theory of finite automata as a formal tool of analysis. Threads directed by conditions on the other hand, seem to be the best informal means for the neuro-biologist to formulate his problems, and then to undertake the investigation by formal means somehow. By thinking in directed threads, moreover, one gets a clear picture of the meaning of the normal form Σ_1^ω for SC-formulae, which is the main objective of the decidability proof, and which is itself relatively easy to decide. The investigation of Σ_1^ω-formulae shows that just the ultimately periodic predicates are definable in SC; thus SC-conditions are good to determine ultimately periodic threads, but no others. In this way the "range of questions" is described, and thus it may be that neuro-biologists and others will be able to translate a good part of their statements into Σ_1^ω-form, and therefore to make good use of the decision procedure in avoiding its most terrifying part.

From a decision procedure mostly follows an account of definability, i.e. a description of the sets and relations, especially functions, definable in models of the theory considered. Such a description is not only of theoretical interest, but gives new information, namely: (a) Information on the formal theory itself; e.g. one can get new normal forms by showing equivalence of classes of formulae. (b) Information on the mathematical content of the theory; e.g. one can get a classification of the models of the theory. It can be very interesting to compare this classification with classical results on the theory, i.e. with the "mathematical" theory not restricted to a fixed logic calculus. (c) Information in turn on the decision procedure; it is easier to understand the steps if one has a concrete imagination of what the particular formulae mean in the model. (d) If the decision procedure is not already effected by a method of quantifier elimination, it may be that one gets conversely such a method by the knowledge on definability.

Investigation of definability along these lines is the main content of chap.II. In II.1.a, following Büchi [2],[3] and Church [8]-[10], it is shown that the restricted recursion formulae of Church, our recursive SC-formulae, and finite automata, all are equivalent in defining sets of words (where a word is a finite part of a thread). - Büchi has shown that just the ultimately periodic sets of natural numbers are definable in SC. We define in §2 the new concept of "fanlike ultimately periodic" relations over natural numbers, and show that just these relations are definable in both SC and the elementary theory CO of congruence and order. It follows that e.g. among the monotonic increasing functions exactly those are definable which are ultimately either constant or a periodic deformation of the identity function. (Thereby the statement of Büchi [2],[3] that any linear function is definable in SC, is corrected.) Further we obtain an effective procedure to eliminate quantifiers from SC-formulae without free predicate variables. Moreover, we prove that one gets a model for SC if one allows instead of arbitrary sets of natural numbers only ultimately periodic sets as interpretations for the predicate variables. At last, in II.3.c we show how to translate the decision procedure for SC from the natural numbers to the integers.

These lecture notes and further results, e.g. on decidable and undecidable extensions of SC (see [35] and [37]), are very much influenced by seminars on "Decidable theories" which were held at Heidelberg by Professor Gert H. Müller and Professor O. Herrmann through several years. I would like to express my thanks to all participants of these seminars. But my deepest thanks are due to Professor Müller himself: most parts of this paper were stimulated by his ideas, and he never hesitated to spend his time to discuss the resulting problems. Especially it was his suggestion to eliminate the use of automata from the decision procedure to make a completeness proof possible; conversely he always insited on the need for smooth concepts to make better understandable both the fact and the proof of the decidability, and for simplification in the procedure to make it applicable. But as a very fact, after so many discussions it is impossible to untie his influence and my own work. I express a special thank to my wife; without her untiring effort in reading and typing so many manuscripts these lecture notes would have never appeared.

Heidelberg, October 27, 1969 Dirk Siefkes

Technical Hints for the Reader

Numbers in square brackets refer to the bibliography at the end of the paper. - Instead of making use of page numbers, we refer to other parts of this paper with the help of chapter, § and section. Thus "theorem I.3.b.3" means "theorem 3 of section b of §3 of chap.I"; if the result referred to is to be found in the same chapter, or §, or section, we delete the corresponding initial part of the code word, thus writing e.g.: lemma 5.c.1, corollary b.2, definition 4. In the same way we refer to sections: I.3.b, 3.b, b. - Throughout the paper we write "DP" and "iff" short for resp. "decision procedure" and "if and only if". Further we use the set-theoretical symbols $\cup, \cap, \{\}, \pi, \subset$ to denote resp. union and intersection of sets, set abstraction, power set, and set inclusion. To enhance clearness we indicate the end of a proof by the sign $\#$. - For all other abbreviations and introduced notations see the "list of symbols and notations" behind the bibliography.