Lecture Notes in Mathematics 1888

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Transseries and Real Differential Algebra

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Foreword

Transseries find their origin in at least three different areas of mathematics: analysis, model theory and computer algebra. They play a crucial role in Écalle's proof of Dulac's conjecture, which is closely related to Hilbert's 16-th problem.

I personally became interested in transseries because they provide an excellent framework for automating asymptotic calculus. While developing several algorithms for computing asymptotic expansions of solutions to non-linear differential equations, it turned out that still a lot of theoretical work on transseries had to be done. This led to part A of my thesis. The aim of the present book is to make this work accessible for non-specialists. The book is self-contained and many exercises have been included for further studies. I hope that it will be suitable for both graduate students and professional mathematicians. In the later chapters, a very elementary background in differential algebra may be helpful.

The book focuses on that part of the theory which should be of common interest for mathematicians working in analysis, model theory or computer algebra. In comparison with my thesis, the exposition has been restricted to the theory of grid-based transseries, which is sufficiently general for solving differential equations, but less general than the well-based setting. On the other hand, I included a more systematic theory of "strong linear algebra", which formalizes computations with infinite summations. As an illustration of the different techniques in this book, I also added a proof of the "differential intermediate value theorem".

I have chosen not to include any developments of specific interest to one of the areas mentioned above, even though the exercises occasionally provide some hints. People interested in the accelero-summation of divergent transseries are invited to read Écalle's work. Part B of my thesis contains effective counterparts of the theoretical algorithms in this book and work is in progress on the analytic counterparts. The model theoretical aspects are currently under development in a joint project with Matthias Aschenbrenner and Lou van den Dries.

The book in its present form would not have existed without the help of several people. First of all, I would like to thank Jean Écalle, for his support and many useful discussions. I am also indoubted to Lou van den Dries and Matthias Aschenbrenner for their careful reading of several chapters and their corrections. Last, but not least, I would like to thank Sylvie for her patience and aptitude to put up with an ever working mathematician.

We finally notice that the present book has been written and typeset using the GNU TeXmacs scientific text editor. This program can be freely downloaded from `http://www.texmacs.org`.

Joris van der Hoeven
Chevreuse 1999–2006

Introduction

The field with no escape

A *transseries* is a formal object, constructed from the real numbers and an infinitely large variable $x \succ 1$, using infinite summation, exponentiation and logarithm. Examples of transseries are:

$$\frac{1}{1-x^{-1}} = 1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots \tag{1}$$

$$\frac{1}{1-x^{-1}-\mathrm{e}^{-x}} = 1+\frac{1}{x}+\frac{1}{x^2}+\cdots+\mathrm{e}^{-x}+2\,\frac{\mathrm{e}^{-x}}{x}+\cdots+\mathrm{e}^{-2x}+\cdots \tag{2}$$

$$\frac{\mathrm{e}^{\frac{x}{1-1/\log x}}}{1-x^{-1}} = \mathrm{e}^{x+\frac{x}{\log x}+\frac{x}{\log^2 x}+\cdots}+\frac{1}{x}\,\mathrm{e}^{x+\frac{x}{\log x}+\frac{x}{\log^2 x}+\cdots}+\cdots \tag{3}$$

$$-\mathrm{e}^x \int \frac{\mathrm{e}^{-x}}{x} = \frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{6}{x^4}+\frac{24}{x^5}-\frac{120}{x^6}+\cdots \tag{4}$$

$$\Gamma(x) = \frac{\sqrt{2\pi}\,\mathrm{e}^{x(\log x-1)}}{x^{1/2}}+\frac{\sqrt{2\pi}\,\mathrm{e}^{x(\log x-1)}}{12\,x^{3/2}}+\frac{\sqrt{2\pi}\,\mathrm{e}^{x(\log x-1)}}{288\,x^{5/2}}+\cdots \tag{5}$$

$$\zeta(x) = 1+2^{-x}+3^{-x}+4^{-x}+\cdots \tag{6}$$

$$\varphi(x) = \frac{1}{x}+\varphi(x^{\pi}) = \frac{1}{x}+\frac{1}{x^{\pi}}+\frac{1}{x^{\pi^2}}+\frac{1}{x^{\pi^3}}+\cdots \tag{7}$$

$$\psi(x) = \frac{1}{x}+\psi(\mathrm{e}^{\log^2 x}) = \frac{1}{x}+\frac{1}{\mathrm{e}^{\log^2 x}}+\frac{1}{\mathrm{e}^{\log^4 x}}+\frac{1}{\mathrm{e}^{\log^8 x}}+\cdots \tag{8}$$

As the examples suggest, transseries are naturally encountered as formal asymptotic solutions of differential or more general functional equations. The name “transseries” therefore has a double signification: transseries are generally *transfinite* and they can model the asymptotic behaviour of *transcendental* functions.

Whereas the transseries (1), (2), (3), (6), (7) and (8) are convergent, the other examples (4) and (5) are divergent. Convergent transseries have a clear analytic meaning and they naturally describe the asymptotic behaviour of

their sums. These properties surprisingly hold in the divergent case as well. Roughly speaking, given a divergent series

$$f = \sum_{n=1}^{\infty} \frac{f_n}{x^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{x^n}$$

like (4), one first applies the formal Borel transformation

$$\hat{f}(\zeta) = (\tilde{\mathcal{B}} f)(\zeta) = \sum_{n=1}^{\infty} \frac{f_n}{(n-1)!}\, \zeta^n = \frac{1}{1+\zeta}.$$

If this Borel transform $\hat{f}$ can be analytically continued on $[0, +\infty)$, then the inverse Laplace transform can be applied analytically:

$$\bar{f}(x) = (\mathcal{L}\hat{f})(x) = \int_0^{\infty} \hat{f}(\zeta)\, \mathrm{e}^{-x\zeta}\, \mathrm{d}x = \int_0^{\infty} \frac{\mathrm{e}^{-x\zeta}}{1+\zeta}\, \mathrm{d}x.$$

The analytic function $\bar{f}$ obtained admits f as its asymptotic expansion. Moreover, the association $f \mapsto \bar{f}$ preserves the ring operations and differentiation. In particular, both f and $\bar{f}$ satisfy the differential equation

$$f' - f = -\frac{1}{x}.$$

Consequently, we may consider $\bar{f}$ as an analytic realization of f. Of course, the above example is very simple. Also, the success of the method is indirectly ensured by the fact that the formal series f has a "natural origin" (in our case, f satisfies a differential equation). The general theory of accelero-summation of transseries, as developed by Écalle [É92, É93], is far more complex, and beyond the scope of this book. Nevertheless, it is important to remember that such a theory *exists*: even though the transseries studied in this this book are purely formal, they generally correspond to genuine analytic functions.

The attentive reader may have noticed another interesting property which is satisfied by some of the transseries (1–8) above: we say that a transseries is *grid-based*, if

GB1. There exists a finite number $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ of infinitesimal "transmonomials", such that f is a multivariate Laurent series in $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$:

$$f = \sum_{\nu_1 \leqslant \alpha_1 \in \mathbb{Z}} \cdots \sum_{\nu_k \leqslant \alpha_k \in \mathbb{Z}} f_{\alpha_1, \ldots, \alpha_k}\, \mathfrak{m}_1^{\alpha_1} \cdots \mathfrak{m}_k^{\alpha_k}.$$

GB2. The property **GB1** is recursively satisfied when replacing f by the logarithm of one of the $\mathfrak{m}_i$.

The examples (1–5) are grid-based. For instance, for (2), we may take $\mathfrak{m}_1 = x^{-1}$ and $\mathfrak{m}_2 = \mathrm{e}^{-x}$. The examples (6–8) are not grid-based, but only *well-based*. The last example even cannot be expanded w.r.t. a finitely generated asymptotic scale with powers in $\mathbb{R}$. As we will see in this book, transseries solutions to algebraic differential equations with grid-based coefficients are necessarily grid-based as well. This immediately implies that the examples (6–8) are

differentially transcendental over $\mathbb{R}$ (see also [GS91]). The fact that grid-based transseries may be considered as multivariate Laurent series also makes them particularly useful for effective computations. For these reasons, we will mainly study grid-based transseries in this book, although generalizations to the well-based setting will be indicated in the exercises.

The resolution of differential and more general equations using transseries presupposes that the set of transseries has a rich structure. Indeed, the transseries form a totally ordered field $\mathbb{T}$ (chapter 4), which is real closed (chapter 3), and stable under differentiation, integration, composition and functional inversion (chapter 5). More remarkably, it also satisfies the differential intermediate value property:

> Given a differential polynomial $P \in \mathbb{T}\{F\}$ and transseries $f < g \in \mathbb{T}$ with $P(f)\,P(g) < 0$, there exists a transseries $h \in \mathbb{T}$ with $f < h < g$ and $P(h) = 0$.

In particular, any algebraic differential equation of odd degree over $\mathbb{T}$, like

$$f^3\,(f')^2\,(f''')^4 + \mathrm{e}^{\mathrm{e}^x}\,f^7 - \Gamma(\Gamma(x\log x))\,f^3\,f' = \log\log x$$

admits a solution in $\mathbb{T}$. In other words, the field of transseries is the first concrete example of what one might call a “real differentially closed field”.

The above closure properties make the field of transseries ideal as a framework for many branches of mathematics. In a sense, it has a similar status as the field of real or complex numbers. In analysis, it has served in Écalle's proof of Dulac's conjecture — the best currently known result on Hilbert's 16-th problem. In model theory, it can be used as a natural model for many theories (reals with exponentiation, ordered differential fields, etc.). In computer algebra, it provides a sufficiently general formal framework for doing asymptotic computations. Furthermore, transseries admit a rich non-archimedean geometry and surprising connections exist with Conway's “field” of surreal numbers.

Historical perspectives

Historically speaking, transseries have their origin in several branches of mathematics, like analysis, model theory, computer algebra and non-archimedean geometry. Let us summarize some of the highlights of this interesting history.

1 Resolution of differential equations by means of power series

It was already recognized by Newton that formal power series are a powerful tool for the resolution of differential equations [New71]. For the resolution of algebraic equations, he already introduced Puiseux series and the Newton polygon method, which will play an important role in this book. During the 18-th century, formal power series were used more and more systematically as a tool for the resolution of differential equations, especially by Euler.

However, the analytic meaning of a formal power series is not always clear. On the one hand side, convergent power series give rise to germs which can usually be continued analytically into multi-valued functions on a Riemann surface. Secondly, formal power series can be divergent and it is not clear *a priori* how to attach reasonable sums to them, even though several recipes for doing this were already known at the time of Euler [Har63, Chapter 1].

With the rigorous formalization of analysis in the 19-th century, criteria for convergence of power series were studied in a more systematic way. In particular, Cauchy and Kovalevskaya developed the well-known majorant method for proving the convergence of power series solutions to certain partial differential equations [vK75]. The analytic continuation of solutions to algebraic and differential equations were also studied in detail [Pui50, BB56] and the Newton polygon method was generalized to differential equations [Fin89].

However, as remarked by Stieltjes [Sti86] and Poincaré [Poi93, Chapître 8], even though divergent power series did not fit well in the spirit of "rigorous mathematics" of that time, they remained very useful from a practical point of view. This raised the problem of developing rigorous analytic methods to attach plausible sums to divergent series. The modern theory of resummation started with Stieltjes, Borel and Hardy [Sti94, Sti95, Bor28], who insisted on the development of summation methods which are stable under the common operations of analysis. Although the topic of divergent series was an active subject of research in the early 20-th century [Har63], it went out of fashion later on.

2 Generalized asymptotic scales

Another approach to the problem of divergence is to attach only an asymptotic meaning to series expansions. The foundations of modern asymptotic calculus were laid by Dubois-Raymond, Poincaré and Hardy.

More general asymptotic scales than those of the form $x^{\mathbb{Z}}$, $x^{\mathbb{Q}}$ or $x^{\mathbb{R}}$ were introduced by Dubois-Raymond [dBR75, dBR77], who also used "Cantor's" diagonal argument in order to construct functions which cannot be expanded with respect to a given scale. Nevertheless, most asymptotic scales occurring in practice consist of so called L-functions, which are constructed from algebraic functions, using the field operations, exponentiation and logarithm. The asymptotic properties of L-functions were investigated in detail by Hardy [Har10, Har11] and form the start of the theory of Hardy fields [Bou61, Ros80, Ros83a, Ros83b, Ros87, Bos81, Bos82, Bos87].

Poincaré [Poi90] also established the equivalence between computations with formal power series and asymptotic expansions. Generalized power series with real exponents [LC93] or monomials in an abstract monomial group [Hah07] were introduced about the same time. However, except in the case of linear differential equations [Fab85, Poi86, Bir09], it seems that nobody had the idea to use such generalized power series in analysis, for instance by using a monomial group consisting of L-functions.

Newton, Borel and Hardy were all aware of the systematic aspects of their theories and they consciously tried to complete their framework so as to capture as much of analysis as possible. The great unifying theory nevertheless had to wait until the late 20-th century and Écalle's work on transseries and Dulac's conjecture [É85, É92, É93, Bra91, Bra92, CNP93].

His theory of accelero-summation filled the last remaining source of instability in Borel's theory. Similarly, the "closure" of Hardy's theory of L-functions under infinite summation removes its instability under functional inversion (see exercise 5.20) and the resolution of differential equations. In other words, the field of accelero-summable transseries seems to correspond to the "framework-with-no-escape" about which Borel and Hardy may have dreamed.

3 Model theory

Despite the importance of transseries in analysis, the first introduction of the formal field of transseries appeared in model theory [Dah84, DG86]. Its roots go back to another major challenge of 20-th century mathematics: proving the completeness and decidability of various mathematical theories.

Gödel's undecidability theorem and the undecidability of arithmetic are well-known results in this direction. More encouraging were the results on the theory of the field of real numbers by Artin-Schreier and later Tarski-Seidenberg [AS26, Tar31, Tar51, Sei54]. Indeed, this theory is complete, decidable and quantifier elimination can be carried out effectively. Tarski also raised the question how to axiomatize the theory of the real numbers with exponentiation and to determine its decidability. This motivated the model-theoretical introduction of the field of transseries as a good candidate of a non-standard model of this theory, and new remarkable properties of the real exponential function were stated.

The model theory of the field of real numbers with the exponential function has been developed a lot in the nineties. An important highlight is Wilkie's theorem [Wil96], which states that the real numbers with exponentiation form an o-minimal structure [vdD98, vdD99]. In these further developments, the field of transseries proved to be interesting for understanding the singularities of real functions which involve exponentiation.

After the encouraging results about the exponential function, it is tempting to generalize the results to more general solutions of differential equations. Several results are known for Pfaffian functions [Kho91, Spe99], but the thing we are really after is a real and/or asymptotic analogue of Ritt-Seidenberg's elimination theory for differential algebra [Rit50, Sei56, Kol73]. Again, it can be expected that a better understanding of differential fields of transseries will lead to results in that direction; see [AvdD02, AvdD01, AvdD04, AvdDvdH05, AvdDvdH] for ongoing work.

4 Computer algebra and automatic asymptotics

We personally became interested in transseries during our work on automatic asymptotics. The aim of this subject is to effectively compute asymptotic expansions for certain explicit functions (such as "exp-log" function) or solutions to algebraic, differential, or more general equations.

In early work on the subject [GG88, Sha90, GG92, Sal91, Gru96, Sha04], considerable effort was needed in order to establish an appropriate framework and to prove the asymptotic relevance of results. Using formal transseries as the privileged framework leads to considerable simplifications: henceforth, with Écalle's accelero-summation theory in the background, one can concentrate on the computationally relevant aspects of the problem. Moreover, the consideration of transfinite expansions allows for the development of a formally exact calculus. This is not possible when asymptotic expansions are restricted to have at most ω terms and difficult in the framework of nested expansions [Sha04].

However, while developing algorithms for the computation of asymptotic expansions, it turned out that the mathematical theory of transseries still had to be further developed. Our results in this direction were finally regrouped in part A of our thesis, which has served as a basis for this book. Even though this book targets a wider public than the computer algebra community, its effective origins remain present at several places: Cartesian representations, the incomplete transbasis theorem, the Newton polygon method, etc.

5 Non-archimedean geometry

Last but not least, the theory of transseries has a strong geometric appeal. Since the field of transseries is a model for the theory of real numbers with exponentiation, it is natural to regard it as a non-standard version of the real line. However, contrary to the real numbers, the transseries also come with a non-trivial derivation and composition. Therefore, it is an interesting challenge to study the geometric properties of differential polynomials, or more general "functions" constructed using the derivation and composition. The differential intermediate value theorem can be thought of as one of the first results in this direction.

An even deeper subject for further study is the analogy with Conway's construction of the "field" of surreal numbers [Con76]. Whereas the surreal numbers come with the important notion of "earliness", transseries can be differentiated and composed. We expect that it is actually possible to construct isomorphisms between the class of surreal numbers and the class of generalized transseries of the reals with so called transfinite iterators of the exponential function and nested transseries. A start of this project has been carried out in collaboration with my former student M. Schmeling [Sch01]. If this project could be completed, this would lead to a remarkable correspondence between growth-rate functions and numbers.

Outline of the contents

Orderings occur in at least two ways in the theory of transseries. On the one hand, the terms in the expansion of a transseries are naturally ordered by their asymptotic magnitude. On the other hand, we have a natural ordering on the field $\mathbb{T}$ of transseries, which extends the ordering on $\mathbb{R}$. In chapter 1, we recall some basic facts about well-quasi-orderings and ordered fields. We also introduce the concept of "asymptotic dominance relations" $\preccurlyeq$, which can be considered as generalizations of valuations. In analysis, $f \preccurlyeq g$ and $f \prec g$ are alternative notations for $f = O(g)$ and $f = o(g)$.

In chapter 2, we introduce the "strong C-algebra of grid-based series" $C[[\mathfrak{M}]]$, where $\mathfrak{M}$ is a so called monomial monoid with a partial quasi-ordering $\preccurlyeq$. Polynomials, ordinary power series, Laurent series, Puiseux series and multivariate power series are all special types of grid-based series. In general, grid-based series carry a transfinite number of terms (even though the order is always bounded by ω^ω) and we study the asymptotic properties of $C[[\mathfrak{M}]]$.

We also lay the foundations for linear algebra with an infinitary summation operator, called "strong linear algebra". Grid-based algebras of the form $C[[\mathfrak{M}]]$, Banach algebras and completions with respect to a valuation are all examples of strong algebras, but we notice that not all strong "serial" algebras are of a topological nature. One important technique in the area of strong linear algebra is to make the infinite sums as large as possible while preserving summability. Different regroupings of terms in such "large sums" can then be used in order to prove identities, using the axiom of "strong associativity". The terms in "large sums" are often indexed by partially ordered grid-based sets. For this reason, it is convenient to develop the theory of grid-based series in the partially ordered setting, even though the ordering $\preccurlyeq$ on transmonomials will be total.

The Newton polygon method is a classical technique for the resolution of algebraic equations with power series coefficients. In chapter 3, we will give a presentation of this method in the grid-based setting. Our exposition is based on the systematic consideration of "asymptotic equations", which are equations with asymptotic side-conditions. This has the advantage that we may associate invariants to the equation like the Newton degree, which simplifies the method from a technical point of view. We also systematically consider derivatives of the equation, so as to quickly separate almost multiple roots.

Chapter 3 also contains a digression on Cartesian representations, which are both useful from a computational point of view and for the definition of convergence. However, they will rarely be used in the sequel, so this part may be skipped at a first reading.

In chapter 4, we construct the field $\mathbb{T} = C[[[x]]]$ of grid-based transseries in x over an "ordered exp-log field" of constants C. Axioms for such constant fields and elementary properties are given in section 4.1. In practice, one usually takes $C = \mathbb{R}$. In computer algebra, one often takes the countable subfield of all "real elementary constants" [Ric97]. It will be shown that $\mathbb{T}$ is again an ordered exp-log field, so it is also possible to take $C = \mathbb{T}$ and construct fields like $\mathbb{R}[[[x]]][[[y]]]$. Notice that our formalism allows for partially defined exponential functions. This is both useful during the construction of $\mathbb{T}$ and for generalizations to the multivariate case.

The construction of $\mathbb{T}$ proceeds by the successive closure of $C[[x^{\mathbb{R}}]]$ under logarithm and exponentiation. Alternatively, one may first close under exponentiation and next under logarithm, following Dahn and Göring or Écalle [DG86, É92]. However, from a model-theoretical point of view, it is more convenient to first close under logarithm, so as to facilitate generalizations of the construction [Sch01]. A consequence of the finiteness properties which underlie grid-based transseries is that they can always be expanded with respect to finite "transbases". Such representations, which will be studied in section 4.4, are very useful from a computational point of view.

In chapter 5, we will define the operations ∂, $\int$, $\circ$ and $\cdot^{\mathrm{inv}}$ on $\mathbb{T}$ and prove that they satisfy the usual rules from calculus. In addition, they satisfy several compatibility properties with the ordering, the asymptotic relations and infinite summation, which are interesting from a model-theoretical point of view. In section 5.4.2, we also prove the Translagrange theorem due to Écalle, which generalizes Lagrange's well-known inversion formula for power series.

Before going on with the study of differential equations, it is convenient to extend the theory from chapter 2 and temporarily return to the general setting of grid-based series. In chapter 6, we develop a "functional analysis" for grid-based series, based on the concept of "grid-based operators". Strongly multilinear operators are special cases of grid-based operators. In particular, multiplication, differentiation and integration of transseries are grid-based operators. General grid-based operators are of the form

$$\Phi(f) = \Phi_0 + \Phi_1(f) + \Phi_2(f, f) + \cdots,$$

where each Φ_i is a strongly i-linear operator. The set $\mathscr{G}(C[[\mathfrak{M}]], C[[\mathfrak{N}]])$ of grid-based operators from $C[[\mathfrak{M}]]$ into $C[[\mathfrak{N}]]$ forms a strong C-vector space, which admits a natural basis of so called "atomic operators". At the end of chapter 6, we prove several implicit function theorems, which will be useful for the resolution of differential equations.

In chapter 7, we study linear differential equations with transseries coefficients. A well-known theorem [Fab85] states that any linear differential equation over $\mathbb{C}[[z]]$ admits a basis of formal solutions of the form

$$(f_0(\sqrt[p]{z}) + \cdots + f_d(\sqrt[p]{z}) \log^d z)\, z^\alpha\, \mathrm{e}^{P(1/\sqrt[p]{z})},$$

with $f_0, \ldots, f_d \in \mathbb{C}[[z]]$, $\alpha \in \mathbb{C}$, $P \in \mathbb{C}[X]$ and $p, d \in \mathbb{N}^>$. We will present a natural generalization of this theorem to the transseries case. Our method is based on a deformation of the algebraic Newton polygon method from chapter 3.

Since the only transseries solution to $f'' + f = 0$ is 0, the solution space of an equation of order r does not necessarily have dimension r. Nevertheless, as will be shown in section 7.7, one does obtain a solution space of dimension r by considering an oscillatory extension of the field of transseries. A remarkable consequence is that linear differential operators can be factored into first order operators in this extension. It will also be shown that operators in $\mathbb{T}[\partial]$ can be factored into first and second order operators.

It should also be noticed that the theory from chapter 7 is compatible with the strong summation and asymptotic relations on $\mathbb{T}$. First of all, the trace T_L of a linear differential operator $L \in \mathbb{T}[\partial]$, which describes the dominant asymptotic behaviour of L, satisfies several remarkable properties (see section 7.3.3). Secondly, any operator $L \in \mathbb{T}[\partial]$ admits a so called distinguished strong right-inverse L^{-1}, with the property that $(L^{-1}\, g)_{\mathfrak{h}} = 0$ when $\mathfrak{h}$ is the dominant monomial of a solution to $L h = 0$. Similarly, we will construct distinguished bases of solutions and distinguished factorizations.

Non-linear differential equations are studied in chapter 8. For simplicity, we restrict our attention to asymptotic algebraic differential equations like

$$P(f) = 0 \qquad (f \prec \mathfrak{v}),$$

with $P \in \mathbb{T}\{F\} = \mathbb{T}[F, F', \ldots]$, but similar techniques apply in more general cases. The generalization of the Newton polygon method to the differential setting contains two major difficulties. First, the "slopes" which lead to the first terms of solutions cannot directly be read off from the Newton polygon. Moreover, such slopes may be due to cancellations of terms of different degrees (like in the usual case) or terms of the same degree. Secondly, it is much harder to "unravel" almost multiple solutions.

In order to circumvent the first problem, we first define the differential Newton polynomial $N_P \in C\{F\}$ associated to the "horizontal slope" (it actually turns out that N_P is always of the form $N_P = Q\,(F')^\nu$ with $Q \in C[F]$). Then the slope which corresponds to solutions of the form $f = c\,\mathfrak{m} + \cdots$ is "admissible" if and only if $N_{P_{\times \mathfrak{m}}}$ admits a non-zero root in C. Here $P_{\times \mathfrak{m}}$ is the unique differential polynomial with $P_{\times \mathfrak{m}}(f) = P(\mathfrak{m}\, f)$ for all f. In section 8.4, we next give a procedure for determining the admissible slopes. The second problem is more pathological, because one has to ensure the absence of iterated logarithms $\log_l = \log \circ \overset{l\times}{\cdots} \circ \log$ with arbitrarily high l in the expansions of solutions. This problem is treated in detail in section 8.6.

The suitably adapted Newton polygon methods allows us to prove several structure theorems about the occurrence of exponentials and logarithms into solutions of algebraic differential equation. We also give a theoretical algorithm for the determination of all solutions.

The last chapter of this book is devoted to the proof the intermediate value theorem for differential polynomials $P \in \mathbb{T}\{F\}$. This theorem ensures the existence of a solution to $P(f) = 0$ on an interval $I = [g, h]$ under the simple hypothesis that P admits a sign-change on I. The main part of the chapter contains a detailed study of the non-archimedean geometry of $\mathbb{T}$. This comprises a classification of its "cuts" and a description of the behaviour of differential polynomials in cuts. In the last section, this theory is combined with the results of chapter 8, and the interval on which a sign-change occurs is shrunk further and further until we hit a root of P.

Notations

A few remarks about the notations used in this book will be appropriate. Notice that a glossary can be found at the end.

1. Given a mapping $f: A_1 \times \cdots \times A_n \to B$ and $S_1 \subseteq A_1, \ldots, S_n \subseteq A_n$, we write
$$f(S_1, \ldots, S_n) = \{f(a_1, \ldots, a_n): a_1 \in S_1, \ldots, a_n \in S_n\}.$$
Similarly, given a set S, we will write $S > 0$ or $S \prec 1$ if $a > 0$ resp. $a \prec 1$ for all $a \in S$. These and other classical notations for sets are extended to families in section 2.4.1.
2. We systematically use the double index convention $(f_i)_j = f_{i,j}$. Given a set $\mathfrak{S}$ of monomials, we also denote $f_{\mathfrak{S}} = \sum_{\mathfrak{m} \in \mathfrak{S}} f_{\mathfrak{m}} \mathfrak{m}$ (this is an exception to the above notation).
3. Given a set S, we will denote by $S^{>}$ its subset of strictly positive elements, $S^{\preccurlyeq}$ its subset of bounded elements, $S^{<,\prec}$ of negative infinitesimal elements, etc. If $S \subseteq C[[\mathfrak{M}]]$ is a set of series, then we also denote $S_{\succ} = \{f_{\succ}: f \in S\}$, where $f_{\succ} = f_{\mathfrak{M}^{\succ}}$, and similarly for $S_{\succcurlyeq}$, $S_{\prec}$, etc. Notice that this is really a special case of notations 1 and 2.
4. Intervals are denoted by (f, g), $(f, g]$, $[f, g)$ or $[f, g]$ depending on whether the left and right sides are open or closed.
5. We systematically denote monomials $\mathfrak{m}, \mathfrak{n}, \ldots$ in the fraktur font and families $\mathcal{F}, \mathcal{G}, \ldots$ using calligraphic characters.

Those readers who are familiar with my thesis should be aware of the following notational changes which occurred during the past years:

Former	$\underline{\prec\!\!\!\prec}$	$\prec\!\!\!\prec$	$\asymp$	$\sim$	$\underline{\prec\!\!\!\prec\!\!\!\prec}$	$\prec\!\!\!\prec\!\!\!\prec$	$\overset{\smile}{\asymp}$		$f^{\uparrow}$	f^{c}	$f^{\downarrow}$
New	$\preccurlyeq$	$\prec$	$\asymp$	$\sim$	$\underline{\prec\!\!\!\prec}$	$\prec\!\!\!\prec$	$\overset{\smile}{\asymp}$	$\overset{\sim}{\approx}$	$f_{\succ}$	$f_{\asymp}$	$f_{\prec}$

There are also a few changes in terminology:

Former	New
normal basis	transbasis
purely exponential transseries	exponential transseries
potential dominant —	starting —
privileged refinement	$\approx$ unravelling