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The Wulff Crystal in Ising and Percolation Models

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To Katti, Benjamin and Nelly

Foreword

Three series of lectures were given at the 34th Probability Summer School in Saint-Flour (July 6–24, 2004), by the Professors Cerf, Lyons and Slade. We have decided to publish these courses separately. This volume contains the course of Professor Cerf. We cordially thank the author for his performance at the summer school, and for the redaction of these notes.

69 participants have attended this school. 35 of them have given a short lecture. The lists of participants and of short lectures are enclosed at the end of the volume.

The Saint-Flour Probability Summer School was founded in 1971. Here are the references of Springer volumes which have been published prior to this one. All numbers refer to the *Lecture Notes in Mathematics* series, except S-50 which refers to volume 50 of the *Lecture Notes in Statistics* series.

1971: vol 307	1980: vol 929	1990: vol 1527	1997: vol 1717
1973: vol 390	1981: vol 976	1991: vol 1541	1998: vol 1738
1974: vol 480	1982: vol 1097	1992: vol 1581	1999: vol 1781
1975: vol 539	1983: vol 1117	1993: vol 1608	2000: vol 1816
1976: vol 598	1984: vol 1180	1994: vol 1648	2001: vol 1837 & 1851
1977: vol 678	1985/86/87: vol 1362 & S-50		2002: vol 1840
1978: vol 774	1988: vol 1427	1995: vol 1690	2003: vol 1869
1979: vol 876	1989: vol 1464	1996: vol 1665	2004: vol 1878 & 1879

Further details can be found on the summer school web site
<http://math.univ-bpclermont.fr/stflour/>

Jean Picard, Université Blaise Pascal
Chairman of the summer school

Preface

This text is a synthesis of recent works aiming at a mathematically rigorous justification of the phase coexistence phenomenon, starting from a microscopic model. It is intended to be self-contained. The proofs which can be found only in research papers have been included, whereas results for which the proofs can be found in classical textbooks are only quoted.

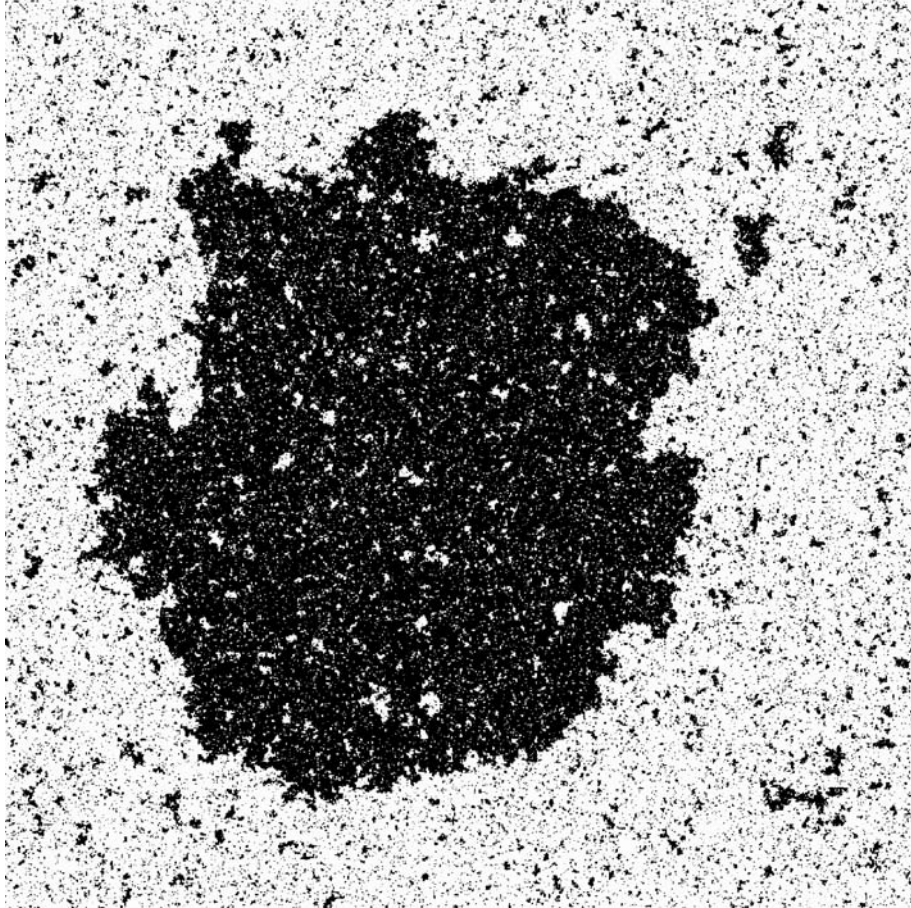
Here is a brief outline of the structure of the text. The main results on the Wulff crystal in Ising and Percolation models are presented in chapter 5. Throughout the text, I focus on three fundamental models: the Bernoulli percolation model, the FK or random cluster model and the Ising model. These models are respectively introduced in chapters 2, 3, 4. The reader interested mainly in the Wulff crystal of the Ising model can proceed directly to section 5.1. Some background on large deviations is provided in chapter 6, and the surface large deviation principles leading to the results of chapter 5 are stated in chapter 7. The associated volume large deviation principles are presented in chapter 8. Chapters 9, 10, 11, 12 contain the fundamental probabilistic estimates for the proofs. The basic geometric tools involved in the proofs are the object of chapters 13, 14, 15. The final steps of the proofs for each model are described in chapters 16, 17, 18, 19.

The simulations are performed with a one Ghz computer running under Linux. The percolation simulations were done with the program `gp-bond` and the Ising simulations with the program `gising`. These programs are available under the GNU General Public License through the web page <http://www.math.u-psud/~cerf>. The pictures of the droplets of oil in the water were taken in my saucepan.

I thank Jean Picard for the efficient and smooth organization of the Saint-Flour summer school. I wish to express my gratitude to Olivier Couronné, Reda Messikh, Katti Millock and Thierry Quentin de Gromard for their help and comments.

Orsay,
October 2005

Raphaël Cerf



Simulation of an Ising Wulff crystal at $T = 2.22$

Contents

Part I Introduction

1	Phase coexistence and subadditivity	3
1.1	Water and oil	3
1.2	Subadditivity	5
1.3	Cramér's theorem	6

Part II Presentation of the models

2	Ising model	15
2.1	Construction of the model	15
2.2	First asymptotics	16
2.3	Phase transition	18
2.4	Proofs of the heuristics	18
3	Bernoulli percolation	25
3.1	The probability space	25
3.2	Order on Ω	26
3.3	Phase transition	26
4	FK or random cluster model	31
4.1	Finite volume FK measures	31
4.2	Phase transition	32
4.3	FK Ising coupling	33
4.4	Boundary conditions	41

Part III Main results

5	The Wulff crystal	45
5.1	Ising model	45
5.2	Bernoulli percolation	55
5.3	FK percolation	58
5.4	What do we know about the Wulff crystal?	60
5.5	Bibliographical comments	62

Part IV Large deviation principles

6	Large deviation theory	67
6.1	Main definitions	67
6.2	I -tightness	68
6.3	Contraction principle	72
6.4	Varadhan's lemma	73
7	Surface large deviation principles	75
7.1	Surface energy	75
7.2	The empirical magnetization	78
7.3	Minimal surfaces	79
7.4	The cluster shapes	82
7.5	FK percolation	83
8	Volume large deviations	85
8.1	Bernoulli percolation	85
8.2	FK percolation	92
8.3	Ising model	96

Part V Fundamental probabilistic estimates

9	Coarse graining	105
9.1	The good blocks	105
9.2	Extension to FK measures	110
9.3	The rescaled lattice	112
9.4	Two rough estimates	114
10	Decoupling	117
10.1	Half-space clusters	117
10.2	Decoupling lemma	121

11 Surface tension	129
11.1 Existence	129
11.2 Finite volume definition	133
11.3 Basic properties	136
11.4 Separating sets	141
11.5 What do we know about the surface tension?	145
12 Interface estimate	147
12.1 Interface lemma	147
12.2 Near the boundary	152
12.3 Percolation setting	153
12.4 Lower bound	155

Part VI Basic geometric tools

13 Sets of finite perimeter	159
13.1 Basic definitions	159
13.2 Covering and differentiating	160
13.3 Caccioppoli sets	164
13.4 Two technical results	167
14 Surface energy	173
14.1 Definition	175
14.2 Lowersemicontinuity and compactness	178
14.3 Covering	178
14.4 Polyhedral approximation	181
15 The Wulff theorem	189
15.1 Statement of the theorem	189
15.2 The anisotropic isoperimetric inequality	190
15.3 The proof of Brothers and Morgan	192
15.4 Stability of the Wulff crystal	197

Part VII Final steps of the proofs

16 LDP for the cluster shapes	203
16.1 Coarse grained image	204
16.2 Exponential contiguity	206
16.3 Local upper bound	209
16.4 Lower bound	211

17 Enhanced upper bound	215
17.1 A lemma from discrete geometry	215
17.2 Uniform large deviation upper bounds	216
17.3 Conclusion of the proof	221
17.4 Extension to FK percolation	227
18 LDP for FK percolation	229
18.1 Coarse grained image	229
18.2 Exponential contiguity	233
18.3 Local upper bound	235
18.4 Lower bound	236
19 LDP for Ising	241
19.1 Coarse grained image	241
19.2 Exponential contiguity	243
19.3 Local upper bound	246
19.4 Lower bound	249
References	253
Index	259
List of participants	261
List of short lectures	263