Lecture Notes in Mathematics 1886

Shigeru Takamura

Splitting Deformations of Degenerations of Complex Curves

Towards the Classification of Atoms of Degenerations, III

Springer

Author

Shigeru Takamura

Department of Mathematics
Graduate School of Science
Kyoto University
Oiwakecho, Kitashirakawa
Sakyo-Ku, Kyoto 606-8502
Japan
e-mail: takamura@math.kyoto-u.ac.jp

Library of Congress Control Number: 2006923235

Mathematics Subject Classification (2000): 14D05, 14J15, 14H15, 32S30

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN-10 3-540-33363-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-33363-0 Springer Berlin Heidelberg New York

DOI 10.1007/b138136

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the authors and SPi using a Springer LaTeX package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11735212 VA41/3100/SPi 5 4 3 2 1 0

Contents

Part II Deformations of Tubular Neighborhoods of Branches

Abstract

This is the third in our series of works which make a systematic study of degenerations of complex curves, and their splitting deformations. The principal aim of the present volume is to develop a new deformation theory of degenerations of complex curves. The construction of these deformations uses special subdivisors of singular fibers, which are characterized by some analytic and combinatorial properties. Intuitively speaking, given a special subdivisor, we will construct a deformation of the degeneration in such a way that the subdivisor is 'barked' (peeled) off from the singular fiber. The construction of these "barking deformations" are very geometric and related to deformations of surface singularities (in particular, cyclic quotient singularities) as well as the mapping class groups of Riemann surfaces (complex curves) via monodromies; moreover the positions of the singularities of a singular fiber appearing in a barking deformation is described in terms of the zeros of a certain polynomial which is expressed in terms of the Riemann theta function and its derivative. In addition to the solid foundation of the theory, we provide several applications, such as (1) a construction of interesting examples of splitting deformations which leads to the class number problem of splitting deformations and (2) the complete classification of absolute atoms of genus from 1 to 5. For genus 1 and 2 cases, this result recovers those of B. Moishezon and E. Horikawa respectively.

Introduction

Wading through,
And wading through,
Yet green mountains still.
(Santoka "Somokuto[1]")

This is the third in our series of works on degenerations of complex curves. (We here use "complex curve" instead of "Riemann surface".) The aim of the present volume is to develop a new deformation theory of degenerations of complex curves. This theory is very geometric and a particular class of subdivisors contained in singular fibers plays a prominent role in the construction of deformations. It also reveals the close relationship between the monodromy of a degeneration and existence of deformations of the degeneration. Moreover, via some diagrams, we may visually understand how a singular fiber is deformed. These deformations are called *barking deformations*, because in the process of deformation, some special subdivisor of the singular fiber looks like "barked" (peeled) off. We point out that barking deformations have a remarkable cross-disciplinary nature; they are related to algebraic geometry, low dimensional topology, and singularity theory.

We will further develop our theory: In [Ta,IV], we describe the vanishing cycles of the nodes of the singular fibers appearing in barking families; we then apply this result to give the Dehn twist decompositions of some automorphisms of Riemann surfaces. In [Ta,V], we develop the moduli theory of splitting deformations, which as a special case, includes the theory of barking deformations over several parameters (in the present volume, we mainly discuss the one-parameter deformation theory).

Background

We will give a brief survey on history and recent development of degenerations of complex curves. Our review is not exhaustive but only covers related topics to our book.

[1] Translated by Hisashi Miura and James Green.

Degenerations of complex curves

A degeneration of complex curves is a one-parameter family of smooth complex curves, which degenerates to a singular complex curve. More precisely, let $\pi : M \to \Delta$ be a proper surjective holomorphic map from a smooth complex surface M to a small disk $\Delta := \{s \in \mathbb{C} : |s| < \delta\}$ such that $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a smooth complex curve of genus g ($g \geq 1$); so the origin $0 \in \Delta$ is the critical value of π. (In what follows, unless otherwise mentioned, complex surfaces (curves) are always supposed to be smooth.) We say that $\pi : M \to \Delta$ is a *degeneration* of complex curves of genus g with the *singular fiber* $X := \pi^{-1}(0)$. For simplicity, we sometimes say "a degeneration of genus g".

Let $f : S \to C$ be a proper surjective holomorphic map from a compact complex surface S to a compact complex curve C, and then S is called a *fibered surface* (e.g. elliptic surface). We note that a degeneration appears as a local model of a fibered surface around a singular fiber: Let X be a singular fiber of $f : S \to C$, and then the restriction of f to a sufficiently small neighborhood (germ) of X in S is a degeneration. To classify fibered surfaces, it is important to understand their local structure — degeneration — around each singular fiber. It is also important to know when the signature $\sigma(S)$ (or some other invariant) of the fibered surface concentrates on singular fibers. Namely, when does the equality $\sigma(S) = \sum_i \sigma_{\mathrm{loc}}(M_i)$ holds?, where M_i is a germ of a singular fiber X_i in S, and $\sigma_{\mathrm{loc}}(M_i)$ denotes the local signature of M_i, and the summation runs over all singular fibers (see a survey [AK]). These questions motivate us to study degenerations and their invariants.

Apart from the (local) signature, we have another basic invariant "monodromy" of a degeneration, which also plays an important role in studying degenerations. Given a degeneration $\pi : M \to \Delta$ of complex curves of genus g, we may associate an element h of the symplectic group $Sp(2g : \mathbb{Z})$ acting on the homology group $H_1(\Sigma_g : \mathbb{Z})$, where Σ_g is a smooth fiber of $\pi : M \to \Delta$. The element h is defined as follows. We take a circle $S^1 := \{\, |s| = r \,\}$ contained in the disk Δ, and then $R := \pi^{-1}(S^1)$ is a real 3-manifold. The map $\pi : R \to S^1$ is a fibration (all fibers are diffeomorphic); that is, R is a Σ_g-bundle over S^1, where Σ_g is a smooth fiber of $\pi : M \to \Delta$. Topologically, R is obtained from a product space $\Sigma_g \times [0,1]$ by the identification of the boundary $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{1\}$ via a homeomorphism γ of Σ_g. We say that γ is the *topological monodromy* of the degeneration $\pi : M \to \Delta$. (It measures how the complex surface M is twisted around the singular fiber X.) Then γ induces an automorphism $h := \gamma_*$ on $H_1(\Sigma_g : \mathbb{Z})$, which is called the *monodromy* of the degeneration. Note that h preserves the intersection form on $H_1(\Sigma_g : \mathbb{Z})$, and so $h \in Sp(2g : \mathbb{Z})$.

Monodromy already appeared in the early study of degenerations, notably the work of Kodaira [Ko1] on the classification of degenerations of elliptic curves (complex curves of genus 1). He showed that there are eight degenerations and determined their monodromies: The singular fibers of eight

degenerations are respectively denoted by $I_n, I_n^*, II, III, IV, II^*, III^*, IV^*$. (Apart from the three types II, III, IV, each corresponds to an extended Dynkin diagram.) Kodaira also gave explicit construction of these eight degenerations.

Subsequently, Namikawa and Ueno [NU] carried out the classification of degenerations of complex curves of genus 2: there are about 120 degenerations. Namikawa and Ueno encountered with new phenomena, which did not occur in the genus 1 case: (1) The topological type of a degeneration is not necessarily determined by its singular fiber: There are topologically different degenerations of complex curves of genus 2 with the same singular fiber. (2) The monodromy does not determine the topological type of a degeneration. In fact, if $g \geq 2$, there are a lot of topologically different degenerations with the trivial topological monodromy. The reason is as follows: The mapping class group MCG_g of a complex curve of genus g has a natural homomorphism $MCG_g \to Sp(2g : \mathbb{Z})$ (homological representation), as $\gamma \in MCG_g$ induces an automorphism γ_* of $H_1(\Sigma_g : \mathbb{Z})$. The kernel of this homomorphism is the *Torelli group* T_g. (Note: If $g = 1$, then T_g is trivial (i.e. the above homomorphism is injective), whereas if $g \geq 2$, then T_g is nontrivial.) In particular, if $g \geq 2$, and the topological monodromy γ of a degeneration belongs to T_g, then $h := \gamma_*$ (monodromy) is the identity.

This fact indicates that monodromy is not powerful enough to classify degenerations. Moreover, as is suggested by Namikawa and Ueno's classification of 120 degenerations of genus 2, there seem a tremendous amount of degenerations of genus g, as g grows higher, and further classifications for genus $3, 4, ...$ got stuck. New development came from topology. Observe that in the converting process from a topological monodromy to a monodromy, some information may be lost, and hence it is natural to guess that a topological monodromy carries more information than a monodromy, and this is the starting point of the work of Matsumoto and Montesinos, which we shall explain. First of all, we note that the topological monodromy of a degeneration is a very special homeomorphism; it is either periodic or pseudo-periodic (see [Im], [ES], [ST]). Here, a homeomorphism γ of a complex curve C is *periodic* if for some positive integer m, γ^m is isotopic to the identity, and *pseudo-periodic* if for some loops (simple closed curves) $l_1, l_2, \ldots, l_n$ on C, the restriction γ on $C \setminus \{l_1, l_2, \ldots, l_n\}$ is periodic. A Dehn twist γ along a loop l on C is an example of a pseudo-periodic homeomorphism, as the restriction of γ to $C \setminus l$ is isotopic to the identity.

Remark 1 There is a classical study of pseudo-periodic homeomorphisms due to Nielsen [Ni1] and [Ni2]; he referred to a pseudo-periodic homeomorphism as algebraically finite type.

For a pseudo-periodic homeomorphism γ, let m be the integer as above, i.e. γ^m on $C \setminus \{l_1, l_2, \ldots, l_n\}$ is isotopic to the identity. Then γ^m is generated by Dehn twists along $l_1, l_2, \ldots, l_n$. According to the direction of the twist, a Dehn twist is called *right* or *left*. A pseudo-periodic homeomorphism γ is *right* or *left* provided that γ^m is generated only by right or left Dehn twists. The complex

structure on a degeneration poses a strong constraint on the property of its topological monodromy. Using the theory of Teichmüller spaces, Earle–Sipe [ES] and Shiga–Tanigawa [ST] demonstrated that any topological monodromy is a right pseudo-periodic homeomorphism — in [MM2], it is called a pseudo-periodic homeomorphism of *negative type*. For example, if the singular fiber is a Lefschetz fiber (a reduced curve with one node), then the topological monodromy is a right Dehn twist along a loop l on a smooth fiber C. Note that the singular fiber is obtained from C by pinching l; in other words, l is the vanishing cycle.

Matsumoto–Montesinos theory

Matsumoto and Montesinos established the converse of the result of Earle–Sipe and Shiga–Tanigawa. Namely, given a periodic or right pseudo-periodic homeomorphism γ, they constructed a degeneration with the topological monodromy γ. Their argument is quite topological, using "open book construction". In [Ta,II], we gave algebro-geometric construction, clarifying the relationship between topological monodromies and quotient singularities.

We denote by $\mathcal{P}_g$ the set of periodic and right pseudo-periodic homeomorphisms of a complex curve of genus g, and denote by $\widehat{\mathcal{P}}_g$ the conjugacy classes of $\mathcal{P}_g$. Next, we denote by $\mathcal{D}_g$ the set of degenerations of complex curves of genus g, and denote by $\widehat{\mathcal{D}}_g$ its topologically equivalent classes. The main result of Matsumoto and Montesinos [MM2] is as follows:

Theorem 2 (Matsumoto and Montesinos [MM2]) *The elements of $\widehat{\mathcal{P}}_g$ are in one to one correspondence with the elements of $\widehat{\mathcal{D}}_g$.*

One important consequence of this theorem is that the topological classification of degenerations completely reduces to the classification of periodic and right pseudo-periodic homeomorphisms.

Matsumoto and Montesinos [MM2] also determined the shape (configuration) of the singular fiber of a degeneration in terms of the data of its topological monodromy — screw numbers and ramification data. Here, we must take care when using the word "shape", because a shape depends on the choice of model of a degeneration, and it changes under blow up or down. Algebraic geometers usually work with the relatively minimal model of a degeneration — a degeneration is *relatively minimal* if any irreducible component of its singular fiber is not an exceptional curve (a projective line with the self-intersection number -1). However, from the viewpoint of topological monodromies, the relatively minimal model is not so natural. The most natural one is the normally minimal model, because it reflects the topological monodromy very well [MM2]. We now review the definition. Express a singular fiber X as a divisor: $X = \sum_i m_i\Theta_i$ where Θ_i is an irreducible component and a positive integer m_i is its multiplicity. Then $\pi : M \to \Delta$ is called *normally minimal* if X satisfies the following conditions:

(1) the reduced curve $X_{\mathrm{red}} := \sum_i \Theta_i$ is normal crossing (i.e. any singularity of X_{red} is a node), and
(2) if Θ_i is an exceptional curve, then Θ_i intersects other irreducible components at at least three points.

We point out that a relatively minimal degeneration, after successive blow up, becomes a normally minimal one, which is uniquely determined from the relatively minimal degeneration.

In what follows, unless otherwise mentioned, we assume that a degeneration is normally minimal. According to whether the topological monodromy is periodic or pseudo-periodic, the singular fiber is *stellar* (star-shaped) or *constellar* (constellation-shaped). Here, a singular fiber X is called stellar[2] if its dual graph is stellar (star-shaped): X has a central irreducible component (*core*), and several chains of projective lines emanating from the core (see Figure 4.2.1, p61). Such a chain of projective lines is called a *branch* of X. A constellar singular fiber is obtained by bonding branches of stellar fibers, and a resulting chain of projective lines after bonding is called a *trunk*; it is a bridge joining two stellar singular fibers.

The number of the singular fibers of genus g increases rapidly, as g grows higher; this is because a constellar singular fiber is constructed from stellar singular fibers in an inductive way with respect to the genus. For instance, a constellar singular fiber of genus 2 is bonding of two stellar singular fibers of genus 1. (Precisely speaking, there is also a constellar singular fiber of genus 2 obtained from one stellar singular fiber of genus 1 by bonding its two branches.) A constellar singular fiber of genus 3 is either bonding of three stellar singular fibers of genus 1, or bonding of two stellar singular fibers of genus 1 and 2. And as g grows, the partition of the integer g increases rapidly, and accordingly the number of constellar singular fibers increases rapidly.

Based on the work of Matsumoto and Montesinos, Ashikaga and Ishizaka [AI] proposed an algorithm to carry out the topological classification of degenerations of given genus. Although the practical computation becomes difficult as genus grows higher, their algorithm settled down the topological classification problem of degenerations at least theoretically. They applied their algorithm to achieve the topological classification for the genus 3 case (see [AI]): The number of degenerations is about 1600, and among them there are about 50 degenerations with stellar singular fibers. (For any genus, the number of stellar singular fibers is much less than that of constellar singular fibers.)

Morsification

There are about 8, 120, and 1600 degenerations of genus 1, 2, and 3 respectively, and as the genus grows higher, the number of degenerations increases

[2] We have a similar notion in singularity theory, that is, a star-shaped singularity: A singularity V is *star-shaped* if the dual graph of the exceptional set in the resolution space of V is star-shaped, e.g. a singularity with $\mathbb{C}^{\times}$-action. See [OW], [Pn].

rapidly. This fact motivates us to consider another kind of classification — "classification of degenerations modulo deformations". Before we explain it, we review related materials from Morse theory, which elucidates the relationship between the shapes of smooth manifolds and smooth functions on them. One of the key ingredients of Morse theory is the Morse Lemma, asserting that we may perturb a smooth function $f : M \to \mathbb{R}$ in such a way that $f_t : M \to \mathbb{R}$ has only non-degenerate critical points. A non-degenerate critical points is stable under arbitrary perturbation, and so the Morse lemma ensures that we may split critical points of f into stable ones under perturbation. Of course, the Morse lemma is a result in the smooth category, but its spirit is carried over to the complex category, for instance, *Morsification of singularities*: When does an isolated singularity V admits a deformation $\{V_t\}$ such that V_t for $t \neq 0$ possesses only A_1-singularities? (It is known that any hypersurface isolated singularity admits a Morsification, e.g. see Dimca [Di] p82)

We next explain *Morsification of singular fibers*, which was advocated by M. Reid [Re]. First of all, we review splitting deformations.

Splitting deformations of degenerations

Let $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \varepsilon\}$ be a sufficiently small disk. Suppose that $\mathcal{M}$ is a complex 3-manifold, and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a proper flat surjective holomorphic map. We set $M_t := \Psi^{-1}(\Delta \times \{t\})$ and $\pi_t := \Psi|_{M_t} : M_t \to \Delta \times \{t\}$. (Hereafter, we denote $\Delta \times \{t\}$ simply by Δ, so that $\pi_t : M_t \to \Delta$.) We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *deformation family* of $\pi : M \to \Delta$ if $\pi_0 : M_0 \to \Delta$ coincides with $\pi : M \to \Delta$. In this case, $\pi_t : M_t \to \Delta$ is referred to as a *deformation* of $\pi : M \to \Delta$.

Suppose that $\pi_t : M_t \to \Delta$ for $t \neq 0$ has at least two singular fibers, say, $X_1, X_2, \ldots, X_n$ $(n \geq 2)$. Then we say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *splitting family* of the degeneration $\pi : M \to \Delta$, and that $\pi_t : M_t \to \Delta$ is a *splitting deformation* of $\pi : M \to \Delta$. In this case, we say that the singular fiber $X = \pi^{-1}(0)$ *splits into* $X_1, X_2, \ldots, X_n$.

To the contrary, if a singular fiber X admits no splitting deformations at all, the degeneration $\pi : M \to \Delta$ is called *atomic*. The singular fiber of the atomic degeneration is called an *atomic fiber*. (Caution: This terminology is not completely rigorous, because a singular fiber does not determine the topological type of a degeneration, so we must use it with care.) A Lefschetz fiber (i.e. a reduced curve with one node) and a multiple $m\Theta$ of a smooth curve Θ, where $m \geq 2$ is an integer, are examples of atomic fibers (see [Ta,I]).

A *Morsification* of a degeneration $\pi : M \to \Delta$ is a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that for $t \neq 0$, all singular fibers of $\pi_t : M_t \to \Delta$ are atomic fibers. Unfortunately this notion is too restrictive, as many degenerations of high genus seem to admit no Morsifications. Instead, we work with a weaker notion "a finite-stage Morsification", defined as follows. If $\pi : M \to \Delta$ is not atomic, take a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, say, X splits into $X_1, X_2, \ldots, X_n$ (the first-stage splitting). If all singular fibers $X_1, X_2, \ldots, X_n$

are atomic, the first-stage splitting is a Morsification. If some X_i is not atomic, then take a sufficiently small neighborhood M_i of X_i in M_t, and then consider the restriction of π_t to M_i, which is a degeneration $\pi_i : M_i \to \Delta$ (called the *fiber germ* of X_i in $\pi_t : M_t \to \Delta$). Next, take a splitting family $\Psi_i : \mathcal{M}_i \to \Delta \times \Delta^\dagger$ of $\pi_i : M_i \to \Delta$, say, X_i splits into $X_{i,1}, X_{i,2}, \dots, X_{i,m}$ (the second-stage splitting). Repeating this process, we finally reach to a set of atomic fibers, say, $X_1', X_2', \dots, X_l'$: Under the finite-stage Morsification, X splits into atomic fibers $X_1', X_2', \dots, X_l'$. In this case, we obtain a smooth 4-manifold M' together with a locally holomorphic map $\pi' : M' \to \Delta$ such that (1) M' is diffeomorphic to M and (2) all singular fibers $X_1', X_2', \dots, X_l'$ of π' are atomic. Here, "locally holomorphic map" means that M' has a complex structure around X_i', and π' is holomorphic with respect to this complex structure. A finite-stage Morsification of a degeneration is useful for studying the topological types of fibered algebraic surfaces.

There is another motivation from algebraic geometry to study Morsification, inspired by the following question: How does an invariant of a degeneration (e.g. local signature, Horikawa index [AA1]) behave under splitting. Specifically, let $\mathrm{inv}(\pi)$ be some invariant of a degeneration $\pi : M \to \Delta$. Suppose that $\pi_t : M_t \to \Delta$ is a splitting deformation, which splits the singular fiber X into singular fibers $X_1, X_2, \dots, X_n$. Then find a formula of the form

$$\mathrm{inv}(\pi) = \sum_{i=1}^{n} \mathrm{inv}(\pi_i) + c,$$

where $\pi_i : M_i \to \Delta$ is a fiber germ of X_i in M_t, and c is a "correction term". For these problems, we refer the reader to excellent surveys [AE], [AK], and also [Re].

A primary concern of the Morsification problem of degenerations is to classify all atomic degenerations. The number of atomic degenerations of genus g must be much less than that of all degenerations of genus g, and so this problem leads us to a reasonable classification — *classification of degenerations modulo deformations.*

When is a degneration atomic? Before we discuss this problem, we explain several methods to construct splitting families.

Double covering method for hyperelliptic degenerations

A hyperelliptic curve C is a complex curve which admits a double covering $C \to \mathbb{P}^1$ branched over $2g+2$ points on $\mathbb{P}^1$, where $g = \mathrm{genus}(C)$. (All complex curves of genus 1 and 2 are hyperelliptic.) A degeneration $\pi : M \to \Delta$ is called *hyperelliptic* provided that any smooth fiber $\pi^{-1}(s)$ is a hyperelliptic curve. In this case, the total space M is expressed as a double covering $M \to \mathbb{P}^1 \times \Delta$ branched over a complex curve (*branch curve*) B in $\mathbb{P}^1 \times \Delta$, and conversely from this double covering, we may recover the hyperelliptic degeneration $\pi : M \to \Delta$. (Precisely speaking, instead of M, we need to take

a (singular) complex surface M' which is bimeromorphic to M.) A deformation B_t ($t \in \Delta^\dagger$) of the branch curve B induces a deformation $M_t \to \mathbb{P}^1 \times \Delta$ (a family of double coverings branched over B_t) of $M \to \mathbb{P}^1 \times \Delta$, which yields a deformation $\pi_t : M_t \to \Delta$ of the degeneration $\pi : M \to \Delta$. If we choose a suitable deformation B_t of the branch curve B, then $\pi_t : M_t \to \Delta$ is a splitting deformation. This construction is called the *double covering method*, originally due to B. Moishezon [Mo] for the genus 1 case; then applied for the genus 2 case by E. Horikawa [Ho], and finally Ashikaga and Arakawa [AA1] generalized to hyperelliptic degenerations of arbitrary genus.

Note that all degenerations of genus 1 and 2 are hyperelliptic, and so the double covering method is powerful for them. However, a complex curve of genus ≥ 3 is not necessarily hyperelliptic. Accordingly, there are non-hyperelliptic degenerations of genus ≥ 3, for which the double covering method cannot be applied.

In this book, we develop a new deformation theory, which is applicable to any degeneration, irrespective of whether it is hyperelliptic or not. Specifically, we introduce the concept of barking deformations of degenerations, and then derive their properties (here "bark" is that of a tree, not that of a dog.)

Barking deformations

The construction of barking deformations is very geometric. In the simplest case, a barking deformation is — intuitively speaking — obtained by barking (peeling) a special subdivisor of the singular fiber from the singular fiber. As applications, we (1) deduce powerful criteria for the splittability of degenerations, (2) provide interesting examples of splitting deformations which lead to the "class number problem" for degenerations, and (3) determine absolute atoms of genus 3,4, and 5. (Genus 1 and 2 case has already been known.)

Now we shall take a close look at topics of this book.

Construction of barking deformations

To simplify the explanation, for the time being, we only consider stellar singular fibers. Recall that a stellar singular fiber has a central irreducible component (a *core*), and chains of projective lines (*branches*) are attached to the core. We express $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$, where Θ_0 is the core with the multiplicity m_0 and $\mathrm{Br}^{(j)}$ is a branch: $\mathrm{Br}^{(j)}$ intersects Θ_0 transversely at one point.

The construction of a barking deformation proceeds as follows. Take a set of special subdivisors (called *crusts*) of the singular fiber X: A crust is a subdivisor contained in X satisfying certain arithmetic and analytic conditions. We then associate the set of crusts with an "initial deformation" around the core. Next, we propagate the initial deformation along all branches of X. Although the propagation is not always possible, if it is possible, we obtain a barking deformation of the degeneration $\pi : M \to \Delta$.

In general, a barking deformation is constructed from a set of crusts. When can we construct a barking deformation from a single crust? For a stellar singular fiber, we may completely answer this question by characterizing such a crust in terms of some arithmetic condition. (This is *not* the case for a constellar singular fiber, which generally has more deformations.) The answer is very simple. The subbranches of such a crust must be one of three *types* A_l, B_l, *and* C_l, and the converse is also valid. For the definition of types A_l, B_l, and C_l, we refer the reader to Definition 9.1.1, p154.

Moreover, we establish the following result (see p283).

Theorem 3 *Let* $\pi : M \to \Delta$ *be a linear degeneration with a stellar singular fiber* X (*see Remark below for "linear degeneration"*). *Suppose that* X *contains a subdivisor* lY *such that* Y *is a crust and any subbranch of* Y *is either of type* A_l, B_l, *or* C_l. *Then* $\pi : M \to \Delta$ *admits a barking family* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *which barks* lY *from* X. *Conversely if a barking family barks a subdivisor* lY *from* X, *then any subbranch of* Y *is either of type* A_l, B_l, *or* C_l.

Remark 4 Roughly speaking, a degeneration is *linear* if for any irreducible component of the singular fiber X, its tubular neighborhood is biholomorphic to its normal bundle. Essentially, we need this assumption only for irreducible components of genus ≥ 2. Indeed, for an irreducible component of genus 0 or 1 with the negative self-intersection number, its tubular neighborhood is always biholomorphic to its normal bundle (Grauert's Theorem [Gr]).

In Theorem 3, the deformation restricted to the tubular neighborhood of a branch of X is also said to be of *type* A_l, B_l, *or* C_l, the type corresponding to that of the subbranch of Y. These three types of deformations possess very beautiful geometric patterns. Among all, type C_l has interesting periodicity (or symmetry). See Figure 12.3.1, p221 for example.

Theorem 3 is generalized to constellar singular fibers as follows (see p332).

Theorem 5 *Let* $\pi : M \to \Delta$ *be a linear degeneration with a constellar singular fiber* X. *Suppose that* X *contains a subdivisor* lY *such that* Y *is a crust and any subbranch and subtrunk of* Y *are either of type* A_l, B_l *or* C_l. *Then* $\pi : M \to \Delta$ *admits a barking family* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *which barks* lY *from* X.
(The converse is not true. See §18.4, p320, and in particular Example 18.4.2.)

Based on this theorem, we introduce an important concept. Let lY be a subdivisor of X such that (1) Y is a crust and (2) any subbranch and subtrunk of Y are either of type A_l, B_l, or C_l. Then we say that Y is a *simple crust* and l is the *barking multiplicity* of Y. Using this terminology, the above theorem is simply stated as: *If a singular fiber contains a simple crust, then the degeneration admits a barking family.* We denote this barking family by $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. For a singular fiber $X_{s,t} := \Psi^{-1}(s,t)$ in $\pi_t : M_t \to \Delta$ ($t \neq 0$), we say that $X_{s,t}$ is the *main fiber* if $s = 0$, and a *subordinate fiber* if $s \neq 0$: The original singular fiber X splits into one main fiber and several subordinate fibers. In §16.4, p288, we describe main and subordinate fibers in

details. It is noteworthy that the main fiber is generally non-reduced (some irreducible component has multiplicity at least 2); whereas each subordinate fiber is reduced, and all singularities on it are A-singularities.

Class number problem for degenerations

Assume that a degeneration $\pi : M \to \Delta$ has two splitting families $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ and $\Psi' : \mathcal{M}' \to \Delta \times \Delta^\dagger$. We say that Ψ and Ψ' are *topologically equivalent* if there exist orientation preserving homeomorphisms $H : \mathcal{M} \to \mathcal{M}'$ and $h : \Delta \times \Delta^\dagger \to \Delta \times \Delta^\dagger$ such that $h(0,0) = (0,0)$ and the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{H} & \mathcal{M}' \\ \Psi \downarrow & & \downarrow \Psi' \\ \Delta \times \Delta^\dagger & \xrightarrow{h} & \Delta \times \Delta^\dagger, \end{array} \qquad \begin{array}{ccc} M_t & \xrightarrow{H_t} & M'_t \\ \pi_t \downarrow & & \downarrow \pi'_t \\ \Delta \times \{t\} & \xrightarrow{h_t} & \Delta \times \{t\}, \end{array}$$

where $H_t := H|_{M_t}$ and $h_t := h|_{\Delta \times \{t\}}$ are restrictions of H and h respectively. (Note: If Ψ and Ψ' are topologically equivalent, then for each t, $\pi_t : M_t \to \Delta$ and $\pi'_t : M'_t \to \Delta$ are topologically equivalent. But the converse is *not* true.) Barking deformations provide interesting examples of topologically different splitting deformations. For instance, we show (see §20.2, p349)

Theorem 6 *Let $\pi : M \to \Delta$ be a degeneration of elliptic curves with the singular fiber II^** (Kodaira's notation [Ko1])*. Then*

(1) *there exist splitting families Ψ and Ψ' that split II^* into III^* and I_1, but Ψ and Ψ' are topologically different, and*
(2) *there exist splitting families Ψ and Ψ' that split II^* into I_3^* and I_1, but Ψ and Ψ' are topologically different.*

Based on this result, we propose the following problem:

Problem 7 (Class number problem for degenerations) *Let $\pi : M \to \Delta$ be a degeneration. Assume that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family of $\pi : M \to \Delta$, which splits X into $X_1, X_2, \ldots, X_n$. Then how many topologically different splitting families that split X into $X_1, X_2, \ldots, X_n$ do there exist?*

(The *class number* of the splitting $X \mapsto X_1, X_2, \ldots, X_n$ is the number of topologically different splitting families that yield this splitting. It is named after the class number of an algebraic number field; roughly, it measures the deviation from unique factorizations of prime ideals.) We will explore this problem in some other paper.

Classification of atomic degenerations

We have another important application of barking deformations, namely, to the classification of atomic degenerations. Recall that a degeneration is *atomic*

provided that it does not admit any splitting family at all. If a singular fiber is either a reduced curve with one node (Lefschetz fiber) or a multiple of a smooth curve, then the degeneration is atomic (see [Ta,I]). This statement is valid regardless to genus, whereas the complete classification of atomic degenerations had been known only for low genus case (genus 1 and 2); the case of genus 1 was done by B. Moishezon [Mo], and that of genus 2 by E. Horikawa [Ho] with some result of Arakawa and Ashikaga [AA1]

Remark 8 [Ho] showed that if a singular fiber of genus 2 is not a Lefschetz fiber, then it splits into singular fibers of type I_1 and type 0, where "type I_1" is a reducible Lefschetz fiber, that is, two elliptic curves intersecting at one point. On the other hand, any singular fiber of type 0 splits into irreducible Lefschetz fibers by Corollary 4.12 of [AA1].

The list of singular fibers of atomic degenerations of genus 1 and 2 is the following:

	atoms
genus 1 (Moishezon [Mo])	$m\Theta$, where $m \geq 2$ and Θ is a smooth elliptic curve, any reduced curve with one node (Lefschetz fiber)
genus 2 (Horikawa [Ho])	any reduced curve with one node (Lefschetz fiber)

What can we say about genus 3 or higher genus case? In [Re] p5, a conjecture due to Xiao Gang is stated:

"A singular fiber X is atomic precisely when X has either a single node, or is a multiple of a smooth curve, or has some other combination of singularities forced by the monodromy, or has a linear system special in the sense of moduli."

M. Reid also conjectured that an atomic fiber of genus 3 is either a Lefschetz fiber (a reduced curve with one node) or a multiple curve 2Θ where Θ is a smooth curve of genus 2.

In [Ta,I] (see §1.2, p30 of this book for the summary), we showed that a degeneration with a constellar singular fiber almost always admits a splitting family. This result is valid for any genus, and so the classification problem reduces to checking the splittability for the 'remaining case' (we explain soon). Before proceeding, we point out that for genus at least 3, there are a lot of degenerations which are topologically equivalent but analytically inequivalent: see Remark below. So, there may be two topologically equivalent degenerations such that one is atomic but another is not. This indicates that for genus at least 3, the notion of atomicness is too strong. We work instead with a weaker notion: "absolutely atomic".

Remark 9 If a singular fiber has an irreducible component, say Θ, of genus at least 2, then the tubular neighborhood of Θ in M is analytically *not* unique. To the contrary, for an irreducible component of genus 0 or 1 with the negative self-intersection number, its tubular neighborhood is always biholomorphic to its normal bundle by Grauert's Theorem [Gr].

A degeneration is called *absolutely atomic* if any degeneration with the same topological type is atomic. So, if a degeneration $\pi : M \to \Delta$ has a topologically equivalent degeneration $\pi' : M' \to \Delta$ that admits a splitting family, then $\pi : M \to \Delta$ is *not* absolutely atomic.

We proposed in [Ta,I]:

Conjucture 10 *A degeneration is absolutely atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

Now we explain our idea to classify absolute atoms. We intend to carry it out by induction on genus. Namely, suppose that Conjecture 10 is valid for genus $\leq g-1$. According to [Ta,I], under this assumption, to classify absolutely atomic degenerations of genus g, we only have to investigate the splittability for degenerations $\pi : M \to \Delta$ such that either

(A) $X = \pi^{-1}(0)$ is stellar, or
(B) X is constellar and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ of multiplicity 1, then Θ is a projective line, and intersects other irreducible components of X only at one point (hence Θ intersects only one irreducible component).

To these cases, we apply Theorems 3 and 5 and their variants (see criteria below). Namely, we try to find a simple crust (or its generalization "a crustal set") of a singular fiber in (A) or (B): See the list of simple crusts for genus ≤ 5 in p487. As a result, we obtain the complete classification of absolute atomic degenerations of genus 3, 4, and 5 as follows.

	absolute atoms
genus 3	2Θ, where Θ is a smooth curve of genus 2, any reduced curve with one node (Lefschetz fiber)
genus 4	3Θ, where Θ is a smooth curve of genus 2, any reduced curve with one node (Lefschetz fiber)
genus 5	4Θ, where Θ is a smooth curve of genus 2, 2Θ, where Θ is a smooth curve of genus 3, any reduced curve with one node (Lefschetz fiber)

This classification also confirms the validity of Conjecture 10 for genus ≤ 5. (For the genus 6 case, we also checked the validity of this conjecture for a large class of degenerations including those with stellar singular fibers.)

We remark that T. Arakawa and T. Ashikaga [AA1], [AA2] classified absolute atoms among degenerations of "hyperelliptic" curves of genus 3; they used the double covering method.

Main criteria for splittability

Now we state our main criteria for splittability. In what follows, unless otherwise mentioned, we assume that degenerations are linear (see Remark 4). First of all, for stellar singular fibers, we shall exhibit criteria which are derived

from Theorem 3. Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X. We denote X by

$$X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)},$$

where $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch (a chain of projective lines) emanating from the core (the central component) Θ_0. See p284 for the following criterion.

Criterion 11 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Then the following statements hold:*

(1) *Suppose that the core Θ_0 is an exceptional curve* (i.e. *Θ_0 is a projective line such that $\Theta_0 \cdot \Theta_0 = -1$*). *Then $\pi : M \to \Delta$ admits a splitting family.*
(2) *Suppose that the core Θ_0 is not an exceptional curve. If X contains a simple crust Y, then $\pi : M \to \Delta$ admits a splitting family.*

(The splitting families in (1) and (2) can be explicitly described.)

See p285 for the following criterion.

Criterion 12 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Set $r = \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$. Suppose that the following conditions* (A) *and* (B) *are satisfied:*

(A) *$N_0 \cong \mathcal{O}_{\Theta_0}(-p_1^{(1)} - p_1^{(2)} - \cdots - p_1^{(r)})$ where N_0 is the normal bundle of Θ_0 in M and $p_1^{(j)} \in \Theta_0$ is the intersection point of Θ_0 and $\mathrm{Br}^{(j)}$,*
(B) *there are r branches among all branches of X, say, $\mathrm{Br}^{(1)}, \mathrm{Br}^{(2)}, \ldots, \mathrm{Br}^{(r)}$, satisfying the following conditions*:
 (B1) *for $j = 1, 2, \ldots, r$, there exists an integer e_j where $1 \le e_j \le \lambda_j$ such that $m_{e_1}^{(1)} = m_{e_2}^{(2)} = \cdots = m_{e_r}^{(r)}$, and*
 (B2) *for $j = 1, 2, \ldots, r$, each irreducible component $\Theta_i^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$ has the self-intersection number -2* (this condition is vacuous for j such that $e_j = 1$).

Then $\pi : M \to \Delta$ admits a splitting family which is explicitly constructed from the above data. (Note: (A) is an analytic condition, while (B) is a numerical one.)

When Θ_0 is a projective line, the above criterion takes a simpler form (see p286):

Criterion 13 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Set $r = \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$. Assume that Θ_0 is a projective line. Suppose that there are r branches among all branches of X, say, $\mathrm{Br}^{(1)}, \mathrm{Br}^{(2)}, \ldots, \mathrm{Br}^{(r)}$, satisfying the following conditions*:

(B1) *for $j = 1, 2, \ldots, r$, there exists an integer e_j where $1 \le e_j \le \lambda_j$ such that $m_{e_1}^{(1)} = m_{e_2}^{(2)} = \cdots = m_{e_r}^{(r)}$,*

(B2) *for* $j = 1, 2, \ldots, r$, *each irreducible component* $\Theta_i^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$ *has the self-intersection number* -2.

Then $\pi : M \to \Delta$ *admits a splitting family.*

We next exhibit splittability criteria for constellar singular fibers (see p293).

Criterion 14 (Trivial Extension Criterion) *Let* X_1 (resp. X_2) *be a stellar singular fiber of* $\pi_1 : M_1 \to \Delta$ (resp. $\pi_2 : M_2 \to \Delta$), *and let* Br_1 (resp. Br_2) *be a branch of* X_1 (resp. X_2). *Let* X *be a constellar singular fiber of* $\pi : M \to \Delta$ *obtained from* X_1 *and* X_2 *by* κ*-bonding of* Br_1 *and* Br_2, *where* κ $(\kappa \geq -1)$ *is an integer.* (Note: Br_1 and Br_2 are joined to become a "κ-trunk" Tk of X. See p293.) *Suppose that* X_1 *contains a simple crust* Y_1 *such that in the case* $\kappa = -1$,

$$\rho(\mathrm{br}_1) + 1 \leq \mathrm{length}(\mathrm{Tk}),$$

where $\rho(\mathrm{br}_1)$ *is the propagation number of the subbranch* br_1 *of* Y_1 *contained in* Br_1 (see (16.4.2), p291). *Then the barking family of* $\pi_1 : M_1 \to \Delta$ *associated with* Y_1 '*trivially*' *extends to that of* $\pi : M \to \Delta$.
(This criterion is easily generalized to the case where X is obtained by bonding an arbitrary number of stellar singular fibers.)

From Criterion 14, for a degeneration with a constellar singular fiber, we may almost always use a simple crust of some stellar singular fiber to construct its splitting family. Thus **the essential part of the classification of absolute atoms reduces to the stellar case** — precisely speaking, there are some exceptional constellar cases which are not covered by Criterion 14.

We note that stellar singular fibers are much fewer than constellar ones. For example, in genus 3 there are about 1600 singular fibers and only about 50 stellar ones among them (see [AI]). We also note that by Criterion 11 (1), if the core of a stellar singular fiber is an exceptional curve, then the singular fiber admits a splitting. Hence we only need to check the splittability of stellar singular fibers whose cores are not exceptional curves — our criteria drastically reduce the number of singular fibers whose splittability must be checked.

Finally we state a very powerful criterion (see p343).

Criterion 15 *Let* $\pi : M \to \Delta$ *be a degeneration of genus* g *with the singular fiber* X. *Then* $\pi : M \to \Delta$ *admits a splitting family if either* (1), (2), *or* (3) *below holds:*

(1) X *contains a simple crust* Y *such that either*
  (1a) Y *contains no exceptional curve, or*
  (1b) *the barking genus* $g_b(Y) \neq g$ (hence $\leq g - 1$).
(2) X *contains an exceptional curve* Θ_0 *such that*
  (2a) *at least one irreducible component of* X *intersecting* Θ_0 *is a projective line, say this component* Θ_1, *and*

(2b) *any irreducible component of X intersecting Θ_0 satisfies the tensor condition with respect to the subdivisor $Y = \Theta_0 + \Theta_1$.*

(3) *X contains an exceptional curve Θ_0 such that any irreducible component intersecting Θ_0 is a projective line.* (Note: If X is stellar, noting that Θ_0 must be the core, this condition is always satisfied.)

Organization of this book

This book is organized as follows. In Part I, after introducing basic definitions, we explain the idea of barking deformations by means of examples without mentioning much theoretical background. We also give instruction on how to draw "figures of deformations", which is extremely useful to understand geometric nature of barking deformations. We hope that Part I gives the reader a perspective of what will be going on. Part II is devoted to detail study of deformations of tubular neighborhoods of branches. Some arithmetic properties of multiplicities are deeply related to the existence of deformations. In Part III, based on the results of Part II, we introduce the notion of barking deformations for degenerations of *compact* complex curves. Theorems 3 and 5 above are proved there. Furthermore we will derive important splittability criteria of singular fibers from these theorems.

In Part IV, we describe the subordinate fibers. We show that the singularities of a subordinate fiber are A-singularities. Moreover, we give the formulas of the number of the singularities on one subordinate fiber as well as the formula of the number of all subordinate fibers in a barking family.

In Part V, we provide the list of representative crusts for a large class of singular fibers of genus from 1 to 5, which is enough for the purpose of classifying absolute atoms. As a consequence we obtain the complete classification of absolute atoms of genus from 1 to 5.

General advice: Most of chapters contain a section which computes the discriminants of deformations — the *discriminant* of a family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a plane curve in $\Delta \times \Delta^\dagger$, given by $D = \{(s,t) \in \Delta \times \Delta^\dagger \;:\; \Psi^{-1}(s,t) \text{ is singular}\}$. This section is slightly technical, and for the first reading, it may be efficient to skip it.

Without figures, it is hard to comprehend or appreciate barking deformations, and for this reason, I included representative figures. I intended to make this book accessible to researchers studying algebraic geometry, low dimensional topology, and singularity theory. I am very happy if I could share my enthusiasm on this subject with the reader.

Acknowledgment. I am extremely indebted to Professors Tadashi Ashikaga, Yukio Matsumoto and Fumio Sakai, to whom I would like to express my deep gratitude. I would like to sincerely thank Professor Oswald Riemenschneider for useful conversations on deformations of singularities. I would

also like to sincerely thank Professors Masanori Ishida, Shoetsu Ogata, Masataka Tomari, Kazushi Ahara, Ikuko Awata, David De Wit, Madoka Ebihara, Toshizumi Fukui, Toru Gocho, Colin Ingalls, Masaharu Ishikawa, Mizuho Ishizaka, Toshio Ito, Yuichi Yamada, for fruitful discussions and comments. I had the opportunity to give lectures on the present work at Saitama University, Tohoku University, and Tokyo University which were of a great help in clarifying my ideas. I would like to thank the audiences there. I also would like to thank the Max-Planck-Institut für Mathematik at Bonn, and the Research Institute for Mathematical Sciences at Kyoto University for hospitality and financial support. This work was partially supported by a grant from JSPS.

Notation

1. $\Delta = \{s \in \mathbb{C} : |s| < \delta\}$ and $\Delta^\dagger = \{t \in \mathbb{C} : |t| < \varepsilon\}$
2. $\mathcal{O}_M$: the sheaf of germs of holomorphic functions on a complex manifold M
3. f_z: the derivative $\dfrac{df}{dz}$ of a function $f(z)$
4. $\mathbb{P}^1$: the projective line (Riemann sphere)
5. For a divisor $D = \sum_i m_i\Theta_i$ on a smooth complex surface,

   $D \geq 0:$ D is a *nonnegative divisor*, i.e. $m_i \geq 0$ for all i

   $D > 0:$ D is an *effective* (or *positive*) *divisor*, i.e. $m_i > 0$ for all i

   $D \geq D':$ $D - D'$ is a nonnegative divisor

   $D_{\mathrm{red}} := \sum_i \Theta_i :$ the underlying reduced curve of D

   $\mathrm{Supp}(D):$ the *support* of D, i.e. D_{red} as a topological space

   We say that D is *connected* if $\mathrm{Supp}(D)$ is connected as a topological space, and that D *intersects* D' at a point p if $\mathrm{Supp}(D)$ intersects $\mathrm{Supp}(D')$ at p.
6. $X = \sum_i m_i\Theta_i$: a singular fiber where m_i is the multiplicity of an irreducible component Θ_i
   $Y = \sum_i n_i\Theta_i$: a subdivisor of X, so n_i satisfies $0 \leq n_i \leq m_i$. Symbolically this condition is expressed by the notation $0 \leq Y \leq X$.
7. $X_{s,t} := \Psi^{-1}(s,t)$: a fiber of a deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$
8. $\Theta_i \cdot \Theta_i$: the *self-intersection number* of Θ_i. A projective line with the self-intersection number $-n$ is called a $(-n)$-*curve*; a (-1)-*curve* is also called an *exceptional curve* (*of the first kind*).
9. $(\Theta_i \cdot \Theta_i)_Y$: the *formal self-intersection number* of Θ_i with respect to a subdivisor Y, p65
10. $\mathrm{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$: an *unfringed branch*,
    $\overline{\mathrm{Br}} = m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$: a *fringed branch* ($m_0\Delta_0$ is a

fringe and Δ_0 is an open disk), p86. Both unfringed branches and fringed branches are often simply called branches.

11. $\overline{\mathrm{br}} := \overline{\mathrm{Br}} \cap Y$: a fringed subbranch p281, contained in a fringed branch $\overline{\mathrm{Br}}$ of a subdivisor Y
12. $X = m_0\Theta_0 + \sum_{j=1}^N \mathrm{Br}^{(j)}$: a *stellar* (star-shaped) singular fiber where Θ_0 is the central component (the *core*) of X, and $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a *branch*
13. For a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^N \mathrm{Br}^{(j)}$,
    - $p_1^{(j)} \in \Theta_0$: the intersection point of the core Θ_0 and a branch $\mathrm{Br}^{(j)}$, i.e. the *attachment point* of a branch to the core
    - N_0 (resp. $N_i^{(j)}$): the normal bundle of Θ_0 (resp. $\Theta_i^{(j)}$) in M
    - σ: the *standard section* of X, which is a holomorphic section of $N_0^{\otimes(-m_0)}$ such that $\mathrm{div}(\sigma) = \sum_{j=1}^N m_1^{(j)} p_1^{(j)}$, i.e. σ has a zero of order $m_1^{(j)}$ at each point $p_1^{(j)}$ $(j = 1, 2, \ldots, N)$
    - $Y = n_0\Theta_0 + \sum_{j=1}^N \mathrm{br}^{(j)}$: a *crust* of X, where $\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}$ is a subbranch of $\mathrm{Br}^{(j)}$
    - τ: a *core section* of a crust Y, which is a meromorphic section of $N_0^{\otimes n_0}$ with a pole of order $n_1^{(j)}$ at $p_1^{(j)}$ $(j = 1, 2, \ldots, N)$
14. $\mathbf{m} = (m_0, m_1, \ldots, m_\lambda)$ for a fringed branch $\overline{\mathrm{Br}} = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$
15. $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ for a fringed subbranch $\overline{\mathrm{br}} = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$
16. DA_e : a *deformation atlas* of length e, p88
17. $DA_{e-1}(Y, d)$: a deformation atlas of length $e-1$ and weight d associated with a subbranch Y of length e, p90
18. The following continued fraction

$$r_1 - \cfrac{1}{r_2 - \cfrac{1}{r_3 - \cfrac{}{\ddots - \cfrac{1}{r_\delta}}}}$$

will be denoted by $r_1 - \frac{1|}{|r_2} - \frac{1|}{|r_3} - \cdots - \frac{1|}{|r_\delta}$.

19. $f_i := f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f_i} := f(z^{p_{i+1}}\zeta^{p_i})$, $(i = 1, 2, \ldots, \lambda)$: a sequence of holomorphic functions associated with a branch $\mathrm{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ and a holomorphic function $f(z)$ (see p106). Here, nonnegative integers $p_0, p_1, \ldots, p_{\lambda+1}$ are inductively defined by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda, \end{cases}$$

where $r_i = -\Theta_i \cdot \Theta_i$ (that is, $-r_i$ is the self-intersection number of Θ_i).

20. ${}_l\mathrm{C}_k$: the number of choices of k elements from the set of l elements, i.e. $\begin{pmatrix} l \\ k \end{pmatrix}$
21. type $B_l^\sharp$: non-proportional type B_l (this notation is used only in tables), p157
22. $\ell(A) := e_1\ell_1 + e_2\ell_2 + \cdots + e_n\ell_n$: the *length* of a waving polynomial

$$A(w, \eta, t) = w^u P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n},$$

where $P_i = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)})$, p186
23. $DA_{e-1}(\mathbf{Y}, \mathbf{d})$: a deformation atlas of weight $\mathbf{d} = \{d_1, d_2, \ldots, d_l\}$ associated with a bunch $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$, p258
24. $\mathrm{div}(\tau) = \sum_i a_i p_i - \sum_j b_j q_j$: the divisor defined by a meromorphic section τ of a line bundle on a complex curve C; τ has a zero of order a_i at p_i and a pole of order b_j at q_j, p266
25. $DA_{\mathbf{e}} = \{\mathcal{W}_0\,,\, DA_{e_j}^{(j)}\}_{j=1,2,\ldots,N}$: a *deformation atlas of size* $\mathbf{e}$ for a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$, where
    (i) $\mathcal{W}_0$ is a deformation of W_0 parameterized by $\Delta \times \Delta^\dagger$, and
    (ii) $DA_{e_j}^{(j)} = \{\mathcal{H}_i^{(j)}\,,\, \mathcal{H}_i^{(j)\prime}\; g_i^{(j)}\}_{i=1,2,\ldots,e_j}$ is a deformation atlas of length e_j for a branch $\mathrm{Br}^{(j)}$ such that under a coordinate change $(z_0, \zeta_0) = (\eta_1^{(j)}, w_1^{(j)})$ around $p_1^{(j)}$, the equation of $\mathcal{W}_0$ becomes that of $\mathcal{H}_1^{(j)}$, p270
26. $D_1 \sim D_2$: two divisors D_1 and D_2 are *linearly equivalent*, p272
27. $\rho(\mathrm{br}^{(j)})$: the *propagation number* of a subbranch $\mathrm{br}^{(j)}$ of type A_l, B_l, or C_l, defined by

$$\rho(\mathrm{br}^{(j)}) = \begin{cases} e+1 & \text{if } \mathrm{br}^{(j)} \text{ is of type } A_l \\ e & \text{if } \mathrm{br}^{(j)} \text{ is of type } B_l \\ f & \text{if } \mathrm{br}^{(j)} \text{ is of type } C_l, \end{cases}$$

where e is the length of $\mathrm{br}^{(j)}$, and for f, see the explanation following (16.4.2), p291
28. $g_b(Y)$: the *barking genus* of a simple crust Y, p295
29. $\mathrm{Tk} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$: an *unfringed trunk*,
    $\overline{\mathrm{Tk}} = m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$: a *fringed trunk* ($m_0\Delta_0$ and $m_{\lambda+1}\Delta_{\lambda+1}$ are *fringes*, and Δ_0 and $\Delta_{\lambda+1}$ are open disks), p310. Both unfringed trunks and fringed trunks are often simply called trunks.
30. $\overline{\mathrm{tk}} := \overline{\mathrm{Tk}} \cap Y$: a fringed subtrunk p330, contained in a fringed trunk $\overline{\mathrm{Tk}}$ of a subdivisor Y
31. $X \to X_1 + X_2 + \cdots + X_n$: A singular fiber X splits into singular fibers $X_1, X_2, \ldots, X_n$, p351