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# Indefinite Inner Product Spaces, Schur Analysis, and Differential Equations

A Volume Dedicated to Heinz Langer

 Birkhäuser

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# Contents

<i>D. Alpay and B. Kirstein</i>	
Editorial Introduction .....	vii

## Part I: Biographical Material

List of Publications of Professor Dr. Dr. h.c. mult. Heinz Langer .....	3
Some Pictures from the Ceremony .....	17
<i>B. Kirstein</i>	
Heinz Langer – Pioneer of Operator Theory in Indefinite Inner Product Spaces .....	23

## Part II: Papers

<i>D.Z. Arov and H. Dym</i>	
The Inverse Monodromy Problem .....	73
<i>J.A. Ball and V. Bolotnikov</i>	
The Bitangential Matrix Nevanlinna–Pick Interpolation Problem Revisited .....	107
<i>J. Behrndt and F. Philipp</i>	
Finite Rank Perturbations in Pontryagin Spaces and a Sturm–Liouville Problem with $\lambda$ -rational Boundary Conditions .....	163
<i>S. Belyi, K.A. Makarov and E. Tsekanovskii</i>	
On Unimodular Transformations of Conservative $L$ -systems .....	191
<i>G. Berschneider and Z. Sasvári</i>	
Spectral Theory of Stationary Random Fields and their Generalizations. A Short Historical Survey .....	217

<i>I. Cho and P.E.T. Jorgensen</i>	
Semicircular-like, and Semicircular Laws Induced by Certain $C^*$ -probability Spaces over the Finite Adele Ring $A_{\mathbb{Q}}$ .....	237
<i>G. Derfel, P.J. Grabner and R.F. Tichy</i>	
On the Asymptotic Behaviour of the Zeros of the Solutions of a Functional-differential Equation with Rescaling .....	281
<i>B. Fritzsche, B. Kirstein and C. Mädler</i>	
Rational $q \times q$ Carathéodory Functions and Central Non-negative Hermitian Measures .....	297
<i>O.O. Ibrogimov and C. Tretter</i>	
On the Spectrum of an Operator in Truncated Fock Space .....	321
<i>S. Hassi, H.S.V. de Snoo and H. Winkler</i>	
Limit Properties of Eigenvalues in Spectral Gaps .....	335
<i>V. Katsnelson</i>	
Self-adjoint Boundary Conditions for the Prolate Spheroid Differential Operator .....	357
<i>L. Klotz and C. Mädler</i>	
An Addendum to a Paper by Li and Zhang .....	387
<i>A. Sakhnovich</i>	
On Accelerants and Their Analogs, and on the Characterization of the Rectangular Weyl Functions for Dirac Systems with Locally Square-Integrable Potentials on a Semi-Axis .....	393
<i>L. Sakhnovich</i>	
Dirac Equation: the Stationary and Dynamical Scattering Problems .....	407
<i>H. de Snoo and H. Woracek</i>	
Compressed Resolvents, $Q$ -functions and $h_0$ -resolvents in Almost Pontryagin Spaces .....	425
<i>F.H. Szafraniec</i>	
Dissymmetrising Inner Product Spaces .....	485

# Editorial Introduction

Daniel Alpay and Bernd Kirstein

*Telle est la morale que Mermoz et d'autres nous ont enseignée. La grandeur d'un métier est peut-être, avant tout, d'unir des hommes: il n'est qu'un luxe véritable, et c'est celui des relations humaines.*

Antoine de Saint-Exupéry, *Terre des hommes* [15, p. 35].<sup>1</sup>

This volume is a tribute to Heinz Langer on the occasion of his eightieth birthday. Two earlier OT volumes (namely, volume 106 and volume 163; see [19] and [35] respectively) were dedicated to Heinz, the first one on the occasion of his sixtieth birthday, and the second one on the occasion of his retirement.

A summary of Heinz' research till the late nineties can be found in the paper [18] by Aad Dijksma and Israel Gohberg, appearing in OT106, and in Aad Dijksma's *Laudatio* [17] in OT163. During the last 20–25 years Heinz was particularly interested in block operator matrices and their special invariant subspaces. These subspaces have applications to the diagonalization of these matrices, and yield also solutions of the corresponding Riccati equations. As a tool for the study of block operator matrices, jointly with C. Tretter the notion of the quadratic numerical range was introduced; see [34]. It turned out to be useful for the localization of the spectrum and other questions, e.g., the factorization of the Schur complement. He also made contributions to the problem of finding the essential spectrum of certain abstract non-selfadjoint  $2 \times 2$  block operator matrices which have found numerous applications to systems of partial differential equations. As

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<sup>1</sup> *This, then, is the moral taught us by Mermoz and his kind. We understand better, because of him, that what constitutes the dignity of a craft is that it creates a fellowship, that it binds men together and fashions for them a common language. For there is but one veritable problem—the problem of human relations.* Translation by Lewis Galantière, see [14, p. 45].

at the beginning of Heinz career, indefinite products are in the center of his interest. One of the key papers of the last 20–25 years is [30] on the existence of a local spectral function. In two papers spectral properties of the Klein–Gordon equation were studied in Pontryagin and Krein spaces; see [31, 32]. We also mention his results on definite type spectra and the (triple) variational principles of operator pencils and analytic operator functions in Hilbert space, where always a Krein space is in the background. Indefinite inner products play also a role in his papers on the spectral theory of Sturm–Liouville equations with singular potentials, or with Weyl–Titchmarsh functions with respect to a singular endpoint: these Weyl–Titchmarsh functions are generalized Nevanlinna functions, a class of functions which was introduced in joint papers with M.G. Krein in the 1970ies; see for instance [23, 24, 25, 26, 27]. The structure of these functions can be found in the papers [16, 20] (see also [12], and for a constructive proof in the scalar case, see [11]), and the structure of the corresponding generalized Schur functions can be found in [23].

As a late consequence of his close collaboration with Krein, in 2015 a joint manuscript with Krein was published which concerns positive definite functions and their continuations from a finite interval to the whole real axis; see [29]. This problem is related to the spectral theory of Sturm–Liouville equations, as the classical moment problems is related to the spectral theory of Jacobi matrices.

The list of publications in OT106 contains 134 entries, and the updated list in OT163 completes the previous list to 171 entries in 2006, while both the current MathScinet listing and the list presented in this volume have over 203 entries, which corresponds to a good 70 papers written over the last 20 years, and shows how productive Heinz is.

The first named editor (DA) worked very hard on Heinz’ papers (and in particular on the above-mentioned series of papers with M.G. Krein) during his doctoral studies [1] at the Weizmann Institute in Israel. Since Heinz was based in East Germany, any contact, even via letters, was in practice impossible, and DA first met Heinz in November 1987 in Groningen, while on a post-doctoral stay under the supervision of Profs. Aad Dijksma and Henk de Snoo. Heinz proposed that Aad, Henk, their student Piet Bruinsma and this editor consider the interpolation problem using Krein’s formula and the theory of resolvent matrices for the description of the self-adjoint extensions of a given Hermitian operator (see [22, 33]); this led in particular to the publications [4, 5]. A bit later, collaboration between Aad, Heinz and DA began (mainly on the Schur algorithm for generalized Schur functions) and lead to seventeen publications, some of them written in collaboration with Thomas Azizov, R. Buursema, Simeon Reich, David Shoikhet, Yuri Shondin, Dan Volok, and Gerald Wanjala; see for instance [2, 3, 8, 9, 10]. The encounter, and the subsequent collaboration with Heinz was fascinating on numerous grounds. To see a world class mathematician at work was a unique experience for a postdoctoral fellow. That Heinz would suggest a problem, and then collaborate, was extraordinary. Numerous subsequent meetings in Beer-Sheva, Berlin, Groningen, Vienna and other places were the occasion of learning the tools of the



trade, both on the technical point of view, and even more importantly, on the human point of view.

The above illustrates a number of qualities of Heinz, his warmth, his unselfishness, his willingness to share his ideas, and help people. In the last ten years or so this editor moved to study stochastic processes and topological vector spaces (*on revient toujours à ses premières amours*) and each time he meets positive definite functions of the form

$$r(t) + \overline{r(s)} - r(t - s)$$

(which play such an important role in various papers of Heinz; see, e.g., [28, 29]), this editor thinks of his debt to Heinz, and he takes this opportunity to thank Heinz for his help and support over the years. Aad Dijksma played a key role in our joint collaboration, and it is a pleasure (and a duty) to thank him too here.

The influence of Heinz is also profound in the second direction of research taken in the last years by the first editor with Fabrizio Colombo and Irene Sabadini from Politecnico di Milano, where in particular Schur analysis is considered in the setting of slice hyperholomorphic functions (see [13] for the latter) and quaternionic Pontryagin spaces. The paper [6] extends the Krein–Langer factorization of generalized Schur functions. The Iohvidov–Krein–Langer book [21] was essential in this paper as well as in the writing of the book [7].

The second named editor (BK) first met Heinz Langer in 1981, when he attended Heinz’ talk at the Congress of Mathematicians of the GDR at Karl-Marx-University Leipzig. This talk considered a topic from the area of Markov processes.

In September 1980, Lutz Peter Klotz began working at the Section Mathematics in Leipzig. Shortly before, he had obtained his doctorate at TU Dresden under Heinz Langer on prediction theory of one-dimensional continuous stationary stochastic processes. At that time, Bernd Fritzsche (BF) and BK worked together on their dissertation on filter theory of multidimensional discrete stationary stochastic processes. Contingent on the natural proximity of their research topics, contact to Lutz Peter Klotz gradually became closer. At this, Lutz Peter naturally told BF and BK more about his supervisor Heinz Langer and his intensive collaboration with M.G. Krein in Odessa. Particularly, they learned that the mathematical interest of Heinz was orientated towards spectral theory of linear operators in spaces with an indefinite metric. But since he held a professorship of stochastics at TU Dresden, he was not allowed to assign dissertation topics on operator theory to graduates of TU Dresden. Those restrictions impeded Heinz’ research to a great extent.

At the final stage of their work on the PhD thesis, BF and BK were keenly interested in presenting their results in the research seminar of Heinz at TU Dresden. They succeeded in this endeavour thanks to the support of Lutz Peter Klotz, who had initiated the contact that finally resulted in their talk in Dresden in 1982.

Heinz Langer then became one of the referees of the doctoral thesis of BF and BK, which they defended in December 1983. Subsequently, the new postdocs



Photograph by Christiane Tretter. All rights reserved.

redefined their direction of research. The main interest became analytical aspects of prediction theory of multidimensional stationary sequences applying methods of Schur analysis. In doing so, they planned to primarily orientate themselves towards contributions of Soviet mathematicians on Schur analysis. Here, the main focus was on the groups built by M.G. Krein in Odessa as well as V.P. Potapov in Charkov. In continuation of this idea, they asked Heinz for his support in establishing direct contacts. He met this wish without hesitation and especially approached D.Z. Arov in Odessa, with whom BF and BK then established intensive collaborations. During a whole series of research stays in Leipzig, D.Z. Arov conveyed essential features of the Adamyan–Arov–Krein theory and its applications to interpolation problems (Nehari interpolation, generalized bitangential Schur–Nevanlinna–Pick interpolation, Nevanlinna–Pick interpolation). D.Z. Arov revealed to BF and BK that he spent plenty of time together with Heinz during his extended stays in Odessa and told about the fast friendship linking Heinz with the mathematicians around M.G. Krein in Odessa.

Since the middle of the 1980s, BF and BK cultivated close scientific contacts to mathematicians from the Soviet Union. One of the first implications of the Perestroika which began to develop under M.S. Gorbachev manifested as numerous invitations extended to Soviet scientists for stays in the GDR being accepted. Starting 1987, BF and BK hosted numerous Soviet mathematicians who devoted themselves to Schur analysis, operator theory, complex analysis and related topics.



Photograph by Christiane Tretter. All rights reserved.

Most of them approached their hosts with the request to visit Heinz Langer at TU Dresden. When this editor got in touch with him in those concrete cases, Heinz always strove to fulfil such a desire. He cordially received the guests coming from Leipzig to Dresden for a brief visit and took plenty of time for scientific discussions.

BK vividly recalls accompanying V.K. Dubovoj for a visit of Heinz Langer at TU Dresden in May 1987. After their scientific discussions, Heinz drove the three of them in his private car to Saxon Switzerland and showed them the Bastei at

Rathen. Particularly memorable to this editor is also the 2nd of October 1989. At that day, he and V.E. Katsnelson went to see Heinz Langer. This was probably one of the last visits of a foreign guest for Heinz Langer during his time in Dresden. On October 7, 1989 (the 40th anniversary of the foundation of the GDR), Heinz started on a journey to Regensburg in the Western part of Germany, from which he would not return to the GDR.

In the week of 15 to 20 October 1989, the INTSEM Schur analysis took place in Leipzig, an international seminar which was supposed to be chaired by Heinz together with Hans-Joachim Girlich (University Leipzig). Aad Dijkstra and Henk de Snoo from Groningen also attended this seminar. Both noticed that opening of the meeting was delayed because the organizers waited for Heinz Langer. After some hesitation, they informed the hosts that Heinz had left the GDR. Unsure about their future without their figurehead, the attendees were shocked and sorrowful. Fortunately, subsequent developments led to a very conciliatory response to these doubts. Heinz' research and scientific collaborations in operator theory have been flourishing ever since he was appointed to a chair at the Technical University of Vienna in 1991 until today. For his scientific achievements he was awarded the Béla-Szökefalvi-Nagy Medal 2009, the Mark-Grigorjewitsch-Krein Prize 2011 and two honorary doctoral degrees from Stockholm University 2015 and from the Technical University of Dresden 2016.

The volume contains bibliographical material (the list of publications of Heinz, the program of the ceremony and the text of the Laudatio of the award of the honorary doctoral degree at Dresden), and sixteen refereed papers written by experts in their fields. These papers can be divided, in a somewhat arbitrary manner, in the following overlapping categories.

**Schur analysis, linear systems and related topics:** In the paper *The inverse monodromy problem* by **D. Arov** and **H. Dym** special classes of monodromy matrices are studied. In the paper *The bitangential matrix Nevanlinna–Pick interpolation problem revisited*, **J. Ball** and **V. Bolotnikov** give a survey of four approaches to classical interpolation problems and study also the case of generalized Schur functions. The paper *On unimodular transformations of conservative  $L$ -systems*, by **S. Belyi**, **K. Makarov** and **E. Tsekanovskii** studies a special family of transformations of linear systems, called unimodular. In the paper *Rational  $q \times q$  Carathéodory functions and central non-negative Hermitian measures* by **B. Fritzsche**, **B. Kirstein** and **C. Mädler** the Riesz–Herglotz measure of a rational  $q \times q$  Carathéodory function is discussed. Furthermore, an application of interpolation to the theory of multivariate autoregressive stationary sequences is given. **A. Sakhnovich** studies in *On accelerants and their analogs and on the characterization of the rectangular Weyl functions for Dirac systems with locally square-integrable potentials on a semi-axis* Dirac systems with a non-square potential. **L. Sakhnovich** studies in *Dirac equation: the stationary and dynamical scattering problems* show that two scattering operators associated to the radial Dirac equation with Coulomb-type potential coincide.



**Differential operators, inverse problems and related topics:** In the paper *On the asymptotic behaviour of the zeros of solutions of one functional-differential equation with rescaling*, by **G. Derfel**, **P. Grabner** and **R. Tichy** the asymptotic behaviour of the solutions of the pantograph equation is considered. **O. Ibrogimov** and **C. Tretter** consider a generalization of the spin-boson model in *On the spectrum of an operator in truncated Fock space*. The paper *Limit properties of eigenvalues in spectral gaps*, by **S. Hassi**, **H. de Snoo** and **H. Winkler** focuses on selfadjoint extensions of linear symmetric operators or relations which admit a spectral gap. Finally, in *Selfadjoint boundary conditions for the prolate spheroid differential operator*, **V. Katsnelson** studies the Fourier operator restricted to a finite symmetric interval.

**Positivity:** The notion of positivity occurs on most, if not all, the papers presented here. It plays a key role in particular in the paper *Spectral theory of stationary random fields and their generalizations. A short historical survey*, by **G. Berschneider** and **Z. Sasvári**. In *An addendum to a paper by Li and Lang*, **L. Klotz** and **C. Mädler** extend a result of Drury on the trace norm and answer a question of Markus and Watkins. In *Dissymmetrising inner product spaces*, **F. Szafraniec** considers what happens when one replaces positivity by a certain duality.

**Pontryagin space setting:** In *Finite rank perturbations in Pontryagin spaces and a Sturm–Liouville problem with  $\lambda$ -rational boundary conditions*, **J. Behrndt** and **F. Philipp** study the resolvent difference of two selfadjoint operators in Pontryagin space and give applications to Sturm–Liouville operators. Finally, **H. de Snoo** and **H. Woracek** consider in *Compressed resolvents,  $Q$ -functions and  $h_0$ -resolvents in almost Pontryagin spaces* the properties of self-adjoint extensions of a symmetric relation.

**Non commutative analysis:** Probability and stochastic processes appear in a number of papers in this volume. The paper *Semicircular-like, and Semicircular Laws Induced by Certain  $C^*$ -Probability Spaces over the Finite Adele Ring  $A_{\mathbb{Q}}$* , by **I. Cho** and **P. Jorgensen** deals with free probability and tensor products.

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