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What is the Genus?

Springer

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To Ghislaine, Fantin and Line

Preface to the English Translation

In France, some students follow special curricula during the first 2 years of their superior formation, in “classes préparatoires.” There, an intensive training is organized for the entrance examinations to teaching institutions in science or engineering, the so-called “grandes écoles.”

Every May, one of those “great schools,” École Polytechnique, organizes a 2-day mathematical conference with lectures given by professional mathematicians and addressed to mathematics teachers of “classes préparatoires.” Each year, the organizers choose a special theme.

In the beginning of 2011, Pascale Harinck, Alain Plagne, and Claude Sabbah invited me to give one of those lectures. The theme of that year was “Histoires de Mathématiques.” This title has an ambiguity in French, as it may be understood both as “History of Mathematics” and “Stories about Mathematics.” I chose to respect this ambiguity by speaking about the history of mathematics and at the same time by telling a story. The subject of this story was suggested to me by Claude Sabbah in his invitation message: “the notion of genus in algebraic geometry, arithmetic and the theory of singularities.”

I accepted because I saw in the genus one of the most fascinating notions of mathematics, in its rich metamorphoses and in the wealth of phenomena it involves. It may be seen as the prototype of the concept of an invariant in geometry. Preparing the talk and writing the accompanying text for the proceedings to be published at the end of the same year appeared to me as an excellent opportunity to learn more about the development of this notion.

At that moment, I could not have imagined that navigating through the original writings of the discoverers would lead me to a book-length text! In it, I followed several of the evolutionary branches of the notion of genus, from its prehistory in problems of integration, through the cases of algebraic curves and their associated Riemann surfaces, then of algebraic surfaces, into higher dimensions. I had of course to omit many aspects of this incredibly versatile concept, but I hope that the reader who follows me will continue this exploration according to her or his own taste.

I am not a professional historian of mathematics, but I love to understand the development of mathematical ideas from this perspective. Such an understanding

seems essential to me both for doing research and for communicating with other mathematicians or with students.

This book is a slightly expanded translation of the original French version [155]. I corrected a few errors; I reformulated several vague sentences; I added some explanations, figures, or references; and I reorganized the index. I also added two new chapters, one about Whitney's work on sphere bundles and another one on Harnack's formula relating the genus of a Riemann surface defined over the reals to the number of connected components of its real locus.

Acknowledgments I took great advantage from the teamwork leading to the book [52], especially the ensuing contact with writings of the nineteenth century. I want to thank all my co-authors. I am also keen to thank Clément Caubel, Youssef Hantout, Andreas Höring, Walter Neumann, Claude Sabbah, Michel Serfati, Olivier Serman, and Bernard Teissier for their help, their remarks, and their advice. I am particularly indebted to Maria Angelica Cueto for her very careful reading of the first version of my English translation and her advice for improving it. I am also very grateful to the language editor Barnaby Sheppard. Finally, I want to thank warmly Ute McCrory for having raised the idea to publish this text as a book in the History of Math subseries of Springer Lecture Notes in Mathematics.

Villeneuve d'Ascq, France Patrick Popescu-Pampu

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Introduction

Nowadays, one of the fastest ways to introduce the mathematical notion of *genus* is probably to say that it is the number of holes of a surface. If one is speaking to a person with enough mathematical education, one has to add that this surface should be compact, connected, orientable, and without boundary. For instance (see Fig. 1), a sphere is of genus 0, a torus is of genus 1, and the surface of a pretzel is of genus 3.

This definition has the advantage of being intuitive: one may explain it through examples even to children. Moreover, with a little training, one can rapidly manage to find the genus of a given surface provided that it is not too twisted or knotted, as in Fig. 2,[1] which shows only surfaces of genus 0, or as in Fig. 3, which shows a surface of genus 5.

The examples of this type enable us to understand that the concept of "hole" is not always meaningful. Is there some other concept, perhaps less intuitive, which could be applied to any surface and which would give the number of holes whenever possible, for instance, for the surfaces of Fig. 1?

Over the last two centuries, many mathematicians have tried to define a concept of "genus" which is applicable to all surfaces, possibly located in spaces of higher dimension, and even to "abstract" surfaces, which are not given inside any ambient space different from themselves.

Let us see how one may arrive at such a definition, which no longer refers to an ambient space. Start from intuitive examples, where the holes are immediately recognizable. Then draw contours which surround those holes on the surface. Since the holes are separated, one may choose those contours to be pairwise disjoint. One comes up with a collection of circles drawn on the surface, exactly as many as the number of holes, as illustrated in Fig. 4.

We have found an idea: draw pairwise disjoint circles on any surface, then count them, and say that their number is the *genus* of the surface. In order to transform this construction into a well-defined concept, one has to explain first under which

[1]Photograph of a runic stone taken in the city of Sigtuna (Sweden) in 1914 by Erik Brate and available at http://commons.wikimedia.org/wiki/File:U_460,_Skr.

Fig. 1 A sphere, a torus, and two pretzels

constraints one must choose the circles and secondly that all such choices give the same number of circles.

One could either choose a single circle, or one could keep drawing circles, each time slightly different from the circles already drawn. In order to understand how to forbid such choices of circles, which would not allow one to arrive at a uniquely defined number, consider again one of the initial examples, in which a circle surrounds each hole. Then cut the surface along these circles. One sees that the new surface remains connected. But, as indicated by as many examples as one desires, adding an extra circle and performing one more cut would disconnect the surface.

One arrives at the following definition:

The genus of a (compact, connected, orientable) surface without boundary is the maximal number of pairwise disjoint circles one can draw on the surface, with connected complement.

It is then a theorem that all the sets of circles which satisfy those constraints have the same number of elements. This definition applies to all abstract surfaces, as it uses only constructions performed inside the surface, without any reference to an ambient space.

Of course, in order to get a definition which is perfectly satisfying not only from the intuitive viewpoint but also logically, one has to define precisely the notions of surface, of circle drawn on it, of cutting along such a circle, and of connectedness. Topology was developed in particular in order to give a meaning to all these concepts. If one then carefully proves the previously stated theorem of

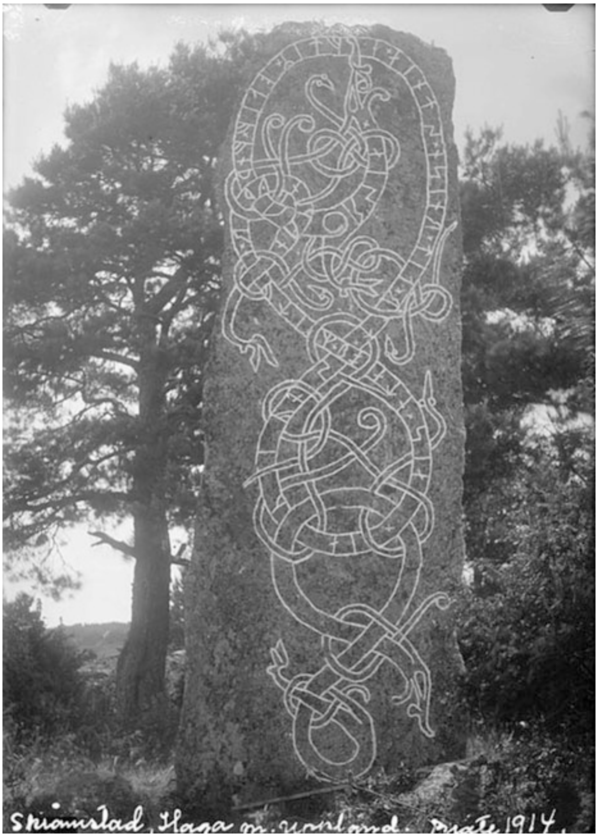

Fig. 2 A runic stone

invariance of the number of circles, one gets indeed a concept of "genus" which is rigorously constructed from a logical viewpoint.

But this does not explain the reason why this concept emerged, nor why it is relevant. In fact, its importance comes from its many avatars, each one of them suggesting other generalizations in higher dimensions, and from the fact that all those generalizations are the basic characteristics used to classify geometric beings in analogy with the classification of living beings.

We will examine here various expressions of this concept during our stroll through time. Exhaustiveness is not an aim of this stroll; it is simply an invitation to listen to the mathematicians of the past. I chose to present many citations, in order to let the actors speak about their motivations and several spectators about their interpretations. In this way, the variety of styles gets emphasized, as well as the evolution of the language, of the questions, and of the viewpoints.

This stroll has three parts: in the first one, we deal with algebraic curves and their topological manifestation once we look at their complex points, forming

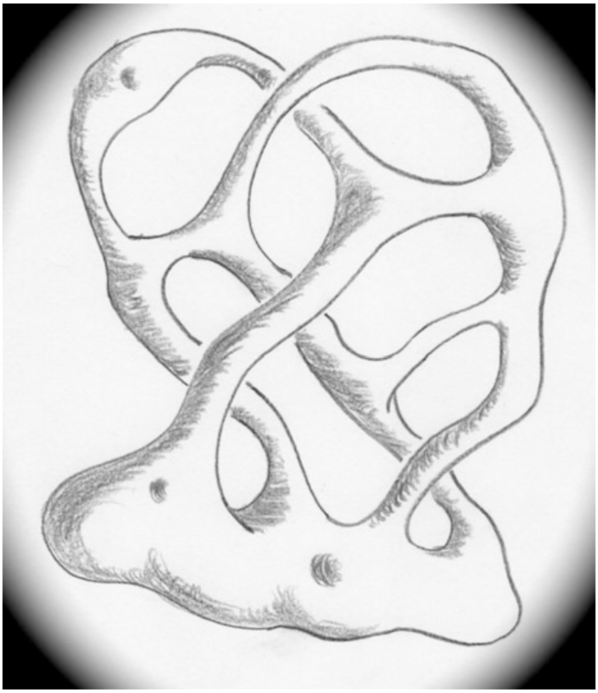

Fig. 3 A knotted surface of genus 5

Fig. 4 Contours surrounding the holes

Riemann surfaces. The second part examines the diverse notions of genus which were introduced for algebraic surfaces. Finally, in the last part, we examine generalizations to arbitrary finite dimensions. But, before starting into this journey, we shall see how Aristotle explained the meaning of the term "$\gamma\acute{\varepsilon}\nu o\varsigma$."