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Mathematical Models of Viscous Friction

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Preface

We present a review of some recent results on the motion of a classical body immersed in an infinitely extended medium and subjected to the action of an external force.

We discuss two cases: when the medium is a gas and when the medium is a fluid. In the first case, the aim is to obtain microscopic models of viscous friction. In the second case, we want to underline some nontrivial features of the motion.

We do not pretend to give a general survey on the subject, but only to discuss some particular results to emphasize the steps done and the open problems. The review is written into the framework of mathematical physics, i.e., it must conjugate the mathematical rigor of the proofs with an explanation of the physical meaning of the results. Sometimes, we study the problem only at a heuristic level, and in this case we emphasize this fact. We give the main ideas and discuss only some aspects of the proofs, while we address to the original papers for the entire technical details and numerical evidences.

A large part of the results presented in this book are the products of fruitful collaborations with several coworkers: K. Aoki, E. Caglioti, S. Caprino, G. Ferrari, F. Manzo, M. Pulvirenti, and T. Tsuji.

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Introduction

The study of the motion of a body immersed in a medium has been a challenging topic since the beginning of science. For many centuries, the correct laws of motion remained unknown. For instance, the empirical observation that a force is needed to maintain a body in a stationary state with nonzero velocity suggested to formulate a law in which such velocity is proportional to the applied force (the so-called Aristotelic Dynamics). After Newton's laws of motion were established, the action of the medium surrounding the body has been summarized in a term called *friction force*. Indeed, the phenomenological observations suggest the introduction of two kinds of forces: the contact friction and the viscous friction. The first one is due to the contact of the body with a solid obstacle, directly or via a thin layer of fluid, and occurs at the level of molecular interactions. Its analysis concerns many interesting topics, not treated in this book. Here, we study the viscous friction, and we analyze in detail some mathematical models.

The search of models of viscous friction is very old. For instance, Newton himself tried to determine the optimal shape of a body to minimize the resistance of the medium. This book is not a review on this topic, too wide to be investigated here. Instead, we focus on some specific features of the problem, with the hope of stimulating the interest of other researchers in this subject.

To be more specific, let us consider the simple case of a body with a fixed shape that moves along the x_1-axis, subjected to an external horizontal force of intensity E and immersed in a homogeneous medium. At a heuristic level, the macroscopic evolution equation reads,

$$\ddot{X}(t) = -G(\dot{X}(t)) + E(X(t)) ,$$

where $X(t)$ is the position of the body at time t, whose mass is assumed to be equal to one, and the *friction term* G is the resultant of all interactions between the body and the medium. The friction term G, usually determined through phenomenological considerations, is assumed to be positive and linear in $V(t) := \dot{X}(t)$ for small velocities of the body. In the simple case of E positive and constant, if $G(V)$ has non-vanishing derivative, the velocity $V(t)$ converges exponentially

fast to a limiting velocity V_∞, which satisfies $G(V_\infty) = E$. In particular, for small intensities of the external force E, one expects a linear response between the external force and the asymptotic velocity (Ohm's law), i.e., $V_\infty = E/c$, where $c > 0$ is the *damping coefficient*.

Of course, it would be desirable to give a microscopic explanation of these facts, where the medium should be described by a system of many particles (atoms or molecules) which interact with the body accelerated by the given field E. Obviously, the behavior of the body will depend on the body/medium interaction. We address the reader to the classical monograph by Landau and Lifshitz[1] for heuristic considerations.

With regard to a mathematically meaningful theory of friction, there are different ways to build a reasonable model of viscous friction. Indeed, besides a faithful microscopic description, one can also approximate the medium either in the framework of kinetic theory with a deterministic and/or stochastic body/medium interaction, or via the fluid mechanics. In this book, we discuss rigorous results and open problems at these three levels of description. We start with the microscopic one, by investigating what we know on a model based on the Newton equations on motion. Then, we discuss some not obvious features of models based on kinetic theory, and we finally show how these features are improved in a model based on fluid mechanics. Now, entering in more detail, we illustrate the content of the different chapters.

In Chap. 1, we tackle the problem at the level of the microscopic description. We consider the perhaps more natural model: the body is represented by a *heavy* point particle subjected to an external constant force, and the medium is composed by many *light* interacting point particles. The whole system evolves according to the laws of classical mechanics, and pairwise conservative forces are assumed to act among the light particles and between the body and the light particles. The medium is supposed to be initially in thermodynamical equilibrium (or in a quite similar state). This very reasonable approach gives rise to a nontrivial problem: to control the time evolution of a system of infinitely many particles. Indeed, to neglect the boundary effects, we are forced to consider an infinitely extended medium. We remark that, although the light particles are initially in thermal equilibrium, the whole system body + background is clearly out of equilibrium. This means that we cannot use statistical properties related to equilibrium, but we must study directly the dynamics of an infinitely extended system. Moreover, it is not enough to know the existence of the time evolution (and also this first step is an important issue, discussed in more detail in Appendix A.1), but we require a good control on the long time behavior of the dynamics. These properties are available solely for systems unbounded in one direction only. We consider systems in which this geometric bound holds.

[1]Landau, L.D., Lifshitz, E.M.: *Physical Kinetics*. Course of Theoretical Physics, Vol. 10. Oxford, New York, Frankfurt: Pergamon Press, 1981.

Of course, this is a limitation, but not very strong, as our goal in this case is at best to establish a *necessary* condition to build a model of viscous friction. More precisely, we show that the interaction body/medium must be singular to have a not vanishing viscous friction. The idea is that, in a good model, a heavy particle pushed by an external force, after a long time, must be slowed down by the medium to reach a finite asymptotic velocity. We show that a bounded body/medium interaction cannot produce such behavior. Indeed, we prove that if the external force is strong enough (with respect to the initial condition of the medium), then the long time motion of the heavy particle is approximately uniformly accelerated. Moreover, in the case of genuinely one-dimensional systems, by using similar (but more complicated) techniques, we can remove the assumption of strong external force, thus showing that the singularity of the body/medium interaction is also a necessary condition for the validity of Ohm's law.

Of course, it would be nice to establish sufficient conditions for this class of models, but at the moment, it seems extremely challenging and it remains an open problem.

How much singular must be the body/medium interaction? Using the Newton's law, the question is too hard for a reason that will be explained later on in the text. For this reason, in Chap. 2 we introduce the mean field approximation for the medium, which is a limit where the mass of the light particles goes to zero, while the number of particles per unit volume diverges, in such a way that the mass density stays finite. This approximation is largely used in physics (see the beginning of Sect. 2.1 for its range of validity, historical notes, references) and gives rise to a reduced description of the medium, encoded by a function $f(\mathbf{x}, \mathbf{v}, t)$ representing the density of mass at time t in the point $(\mathbf{x}, \mathbf{v})$ of the one-particle phase space. The time evolution of f is governed by an integro-differential equation, usually referred to as the Vlasov equation.

Such approximation is quite appropriate in our context, as the ratio between the mass of the body and the mass of a particle of the medium is very large. (Actually, in the main part of this book, we use the mean field approximation for a free gas, but we hope it is possible to extend the analysis to the case of an interacting Vlasov system.) A semi-heuristic argument then suggests that a reasonable model of viscous friction must have a not integrable body/medium force. Remarkably, this conjecture implies that the Coulomb interaction is not singular enough to give rise to a viscous friction, in agreement with the *runaway effect*, experimentally observed in plasma physics.[2]

In Chap. 3, we arrive to a reasonable model of viscous friction. The simplest model to consider is a gas of free light particles elastically interacting with a rigid body. This kind of interaction gives rise to a very irregular motion, with fluctuations that are very small if the ratio between the mass of the body and that of the gas particles is very large. Such issue has been approached more than a century ago,

[2] Ibid.

in the seminal contributions of Einstein[3] and Smoluchowski,[4] which aimed at a microscopic explanation of Brownian motion. In spite of the fluctuations caused by collisions, the averaged motion of the body is expected to be regular and sufficient to give a correct description of the macroscopic behavior of the system. To avoid the difficulties connected with the computations of the averaged quantities, we can alternatively consider the gas in the mean-field approximation discussed before.

In this approximation, we prove that the body reaches an asymptotic state with a law that depends on the temporal correlations and, in general, needs a very long time to be reached. We remark that if one neglects the recollisions, the approach to this asymptotic state is exponential in time (a part some particular cases). It is usual in physics to simulate the viscous friction as the effect of many deterministic or stochastic hits of particles that disappear after the collision. But if this schematic model is not valid and a particle of the medium hits the body twice (or many times), the convergence in time changes drastically and it follows a power law. We show that the form of this law is not universal, but it depends on the shape of the body (convex or concave) and on the nature of the hit (elastic or diffusive). We also discuss a case in which the body is not rigid but elastic, and we quote there numerical works that confirm these statements.

Of course, the relevance of the temporal correlations depends on the physical parameters of the problem, and it is larger at low temperature. (The mean velocity of the particles of the medium is proportional to the square root of the temperature and this value must be compared with the velocity of the body.)

It is an open problem to perform a similar analysis in the presence of an interaction among the particles of the medium, that could drastically change the asymptotic behavior of the gas. Moreover, instead of an interacting Vlasov system, one could consider a medium described via other kinetic equations such as the Boltzmann equation. We have not analytical results in this direction, but in Chap. 3 we quote a numerical study on this topic.

In Chap. 4, we discuss a case in which the medium is described as a fluid where the temporal correlations are very large: a body moving in an incompressible fluid in the Stokes approximation. As we can expect (and in part it is already known), the approach in time to the asymptotic state is very slow, more than that found in the previous case.

It is worthwhile to remark here a well-known fact, perhaps not enough emphasized in the basic courses in physics: in this fluid approximation, the friction force is proportional to the velocity in a stationary state (Stokes law), but in general, when the state is not stationary, memory terms are present. Therefore, to approximating the viscous friction as a term proportional to the velocity alone is reasonable only if one takes into account that there are two different scales of time: a small scale,

[3]Einstein, E.: Ueber die von der molekularkinetischen theorie der waerme geforderte bewegung von in ruhenden fluessigkeiten suspendierten teilchen. Ann. Phys., NY **17** (1905).

[4]Smoluchowski, M.: Zur kinetischen theorie der Brownschen molekularbewegung und der suspensionen. Ann. Phys., NY **21** (1906).

where the convergence to the asymptotic velocity is power law, and a large scale, where such convergence is exponential in time.

Finally, in Appendix A we review the principal results on the Hamiltonian evolution of infinitely extended system. For the convenience of the reader, it is self-contained and independent of the rest of the book.

A warning for the reader: because of the mathematical nature of the book, we must give (or sketch) the proofs of our statements and some of them are quite technical. In a first reading, it is possible to jump the proofs and to go back to their study later on.

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