Lecture Notes in Mathematics 2131

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Approaching the Kannan-Lovász-Simonovits and Variance Conjectures

Springer

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ISBN 978-3-319-13262-4 ISBN 978-3-319-13263-1 (eBook)
DOI 10.1007/978-3-319-13263-1
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2014959425

Mathematics Subject Classification (2010): 46Bxx, 52Axx, 60-XX, 28Axx

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Asymptotic geometric analysis is a rather new branch of research in mathematics, coming from modern functional analysis and, more specifically, from the local theory of Banach spaces when it interplays with classical convex geometry and probability which studies high dimensional phenomena.

It seems to us that the starting point of the theory can be found in Milman's approach to the proof of Dvoretzky's theorem [9]: "Given any norm in R^n, for a random $C \log n$-dimensional subspace of R^n ($C > 0$ is an absolute constant) the norm is almost Euclidean". In a geometrical setting, "given any convex body in R^n, a random $C \log n$-dimensional section of it is almost a Euclidean ball".

It was well known from the very beginning of the theory that many infinite-dimensional Banach spaces cannot have infinite-dimensional closed subspaces isomorphic to the Hilbert space. Thus, Dvoretzky's theorem broke down the idea that the theory of finite-dimensional normed spaces could be a good approach for the study of the infinite-dimensional ones. On the contrary, when we are working on R^n and n is increasing to infinity, new and unexpected properties appear in the spaces and that was the blowup of a new theory, asymptotic in essence.

Officially, asymptotic geometric analysis was born at the end of the last century and is now growing very fast since it is connected with many other parts of mathematics and also of mathematical physics and theoretical computer sciences. Nowadays the achievements of asymptotic geometric analysis show new and unexpected phenomena for high dimensions, which occur in several domains of mathematics and other sciences when we are dealing with a huge number of variables. It also includes connections with asymptotic combinatorics, complexity of graph theory and random matrices. Furthermore, in its development asymptotic geometric analysis uses tools coming from harmonic analysis, PDEs, Riemannian geometry, information and learning theories, quantum versions of this, and some other areas in mathematics. We can find several books concerning or related to these topics. For instance, in the very origins [9–11], and more recently [2, 3, 6, 7], and some others, especially several numbers of the collection Lecture Notes in Mathematics (Springer) entitled "Geometric Aspects of Functional Analysis", Israel Seminar, GAFA (see the references in [5], [8], and [4]).

The readers interested in an overview of the whole theory should read the very recent book by Brazatikos, Giannopoulos, Valettas, and Vritsiou entitled "Geometry of Isotropic Log-Concave Measures" [4], in which the state of the art of the theory is perfectly explained. Our lecture notes focus only on a tiny part of this new theory. They concern two main conjectures: the Kannan-Lovász-Simonovits (KLS) spectral gap and the variance conjectures. Obviously, this is a small fragment of this theory, but it has become very active in research. Possibly, many readers will miss the hyperplane conjecture or slicing problem in these lecture notes. Certainly, this third and very important conjecture also appears in the notes, although only in a tangential manner in order to show its relation with the other two conjectures.

As we commented before, the monograph [4] is really the state of the art in asymptotic geometric analysis. Since the advances in the theory are already gathered there, we feel that we must explain the reader what are the differences between what he will find in these notes and what he can find in the aforementioned monograph.

In [4], complete information on all the topics in the theory can be found, with detailed proofs of each result. Many of the facts appear with the original approach in the corresponding citations and some others with new proofs and improvements. Our approach is aimed by a different intention. We want to present the theory in such a way that it allows interested people, even the ones who are not experts in the field, to get a quick account on the treated topics. In order to do it, we intend to go directly to the core of these two problems, simplifying the exposition in some cases and, in some other cases, offering a presentation of the methods suitable for the professionals with not much background in analysis, geometry, or probability. We expect that both experts and the less initiated, professional researchers interested in other different subjects as well as graduate student in mathematics, can get directly into these topics in the theory without any special effort. This is our main reason to present the theory, avoiding the need of a deep knowledge of the modern theory of convex bodies. Our work goes directly to connect isoperimetric-type inequalities and functional inequalities to offer the interested reader a fast approach to the center of the Kannan-Lovász-Simonovits and variance conjectures, which we think are very natural, modern, and interesting problems.

We also try to complete the information related to these conjectures appearing in the reference quoted before by adding some special examples which do not appear in it. These are some of the contents in Chap. 2.

In the same spirit we include some topics in Chap. 1, Sect. 1.7.2, corresponding to the case in which the probability we are working with is not isotropic, since it is not clear at this moment, as far as we know, how to pass from the variance of a function for an isotropic measure to the corresponding one for any of its linear deformations.

These lecture notes are divided into three chapters plus an appendix. Let us comment the contents in these final lines of the Preface.

Chapter 1 is composed of seven sections. The first four are introductory, introduce the conjectures, and their connection with theoretical computer science. Section 1.5 is dedicated to give the theorem by E. Milman on the role of convexity

in the isoperimetry for log-concave probabilities. The two conjectures, KLS and variance, are presented in the last two sections of this first chapter.

Chapter 2, composed of four sections, is dedicated to present the main examples where one or both conjectures are known to be true. The known examples of uniform probabilities on convex bodies which verify the KLS conjecture are some revolution bodies, the simplex and the ℓ_p^n-balls, $1 \leq p \leq \infty$. The fourth section develops Klartag's results for unconditional log-concave probabilities. We also study the negative square correlation property and the examples satisfying it.

In Chap. 3 we present four important results in this theory. The first two, the theorems by Eldan–Klartag and Ball–Nguyen, relate the variance or the KLS conjectures, respectively, with the hyperplane conjecture.

Next we offer an approach to Eldan's work on the relation between the thin-shell width and the KLS conjecture. Eventually we present the main ideas to prove the best known estimate for the thin-shell width given by Guédon–Milman. We want to mention here that we will only sketch the proofs in this chapter, since our intention is to offer the main ideas and give the results in a compelling way.

In the appendix we present some basic facts related to Prékopa–Leindler, Brunn–Minkowski, and Borell's inequalities.

A part of these notes has been explained by the second author in the "VI International Course of Mathematical Analysis in Andalucía", held in Antequera in September 2014 [1].

We would like to finish this preface thanking Prof. Julio Bernués for many discussions that helped us improving the presentation of these notes, Prof. Darío Cordero–Erausquin for providing us his notes on "La preuve de Eldan–Klartag un peu allegée" and allowing us to reproduce them here, and the anonymous referees for many useful comments that helped us to improve the final presentation of this monograph.

This work has been done with the financial support of MTM2013-42105-P and DGA E-64 projects and of Institut Universitari de Matemàtiques i Aplicacions de Castelló.

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October 2014

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