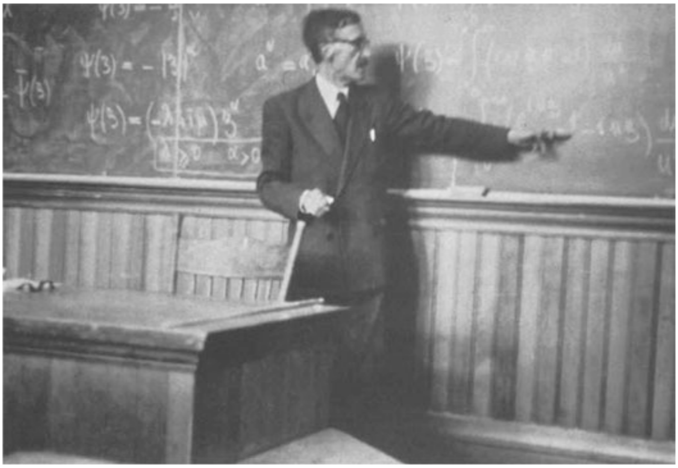
Lecture Notes in Mathematics 2128



"Lévy Matters" is a subseries of the Springer Lecture Notes in Mathematics, devoted to the dissemination of important developments in the area of Stochastics that are rooted in the theory of Lévy processes. Each volume will contain state-of-the-art theoretical results as well as applications of this rapidly evolving field, with special emphasis on the case of discontinuous paths. Contributions to this series by leading experts will present or survey new and exciting areas of recent theoretical developments, or will focus on some of the more promising applications in related fields. In this way each volume will constitute a reference text that will serve PhD students, postdoctoral researchers and seasoned researchers alike.

The volumes in this subseries are published under the auspices of the Bernoulli Society.

Denis Belomestny • Fabienne Comte •
Valentine Genon-Catalot • Hiroki Masuda •
Markus Reiß

Lévy Matters IV

Estimation for Discretely Observed Lévy Processes

Springer

Denis Belomestny
Faculty of Mathematics
Duisburg-Essen University
Thea-Leymann-Str. 9, D-45127
Essen, Germany
and
National University Higher School of Economics
Moscow, Russia

Fabienne Comte
Valentine Genon-Catalot
MAP5, UMR CNRS 8145
University Paris Descartes, Sorbonne Paris Cité
Paris, France

Hiroki Masuda
Institute of Mathematics for Industry
Kyushu University
Fukuoka, Japan

Markus Reiß
Institut für Mathematik
Humboldt-Universität zu Berlin
Berlin, Germany

ISBN 978-3-319-12372-1 ISBN 978-3-319-12373-8 (eBook)
DOI 10.1007/978-3-319-12373-8
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

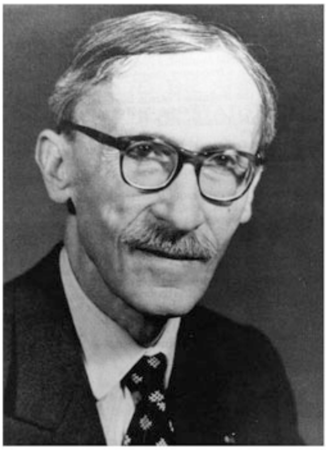
Library of Congress Control Number: 2014958018

Mathematics Subject Classification (2010): 60G10, 60G70, 60J10, 62G05, 62M05, 60F05, 62F12, 60G51, 60G52, 60G18, 91B28, 91B84

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface to the Series Lévy Matters



©: Bildarchiv des Mathematischen
Forschungsinstituts Oberwolfach

Over the past 10–15 years, we have seen a revival of general Lévy processes theory as well as a burst of new applications. In the past, Brownian motion or the Poisson process had been considered as appropriate models for most applications. Nowadays, the need for more realistic modelling of irregular behaviour of phenomena in nature and society like jumps, bursts and extremes has led to a renaissance of the theory of general Lévy processes. Theoretical and applied researchers in fields as diverse as quantum theory, statistical physics, meteorology, seismology, statistics, insurance, finance and telecommunication have realized the enormous flexibility of Lévy models in modelling jumps, tails, dependence and sample path behaviour. Lévy processes or Lévy-driven processes feature slow or rapid structural breaks, extremal behaviour, clustering and clumping of points.

Tools and techniques from related but distinct mathematical fields, such as point processes, stochastic integration, probability theory in abstract spaces and differential geometry, have contributed to a better understanding of Lévy jump processes.

As in many other fields, the enormous power of modern computers has also changed the view of Lévy processes. Simulation methods for paths of Lévy processes and realizations of their functionals have been developed. Monte Carlo simulation makes it possible to determine the distribution of functionals of sample paths of Lévy processes to a high level of accuracy.

This development of Lévy processes was accompanied and triggered by a series of Conferences on Lévy Processes: Theory and Applications. The First and Second Conferences were held in Aarhus (1999, 2002), the Third in Paris (2003), the Fourth in Manchester (2005) and the Fifth in Copenhagen (2007).

To show the broad spectrum of these conferences, the following topics are taken from the announcement of the Copenhagen conference:

- Structural results for Lévy processes: distribution and path properties
- Lévy trees, superprocesses and branching theory
- Fractal processes and fractal phenomena
- Stable and infinitely divisible processes and distributions
- Applications in finance, physics, biosciences and telecommunications
- Lévy processes on abstract structures
- Statistical, numerical and simulation aspects of Lévy processes
- Lévy and stable random fields

At the Conference on Lévy Processes: Theory and Applications in Copenhagen-the idea was born to start a series of Lecture Notes on Lévy processes to bear witness of the exciting recent advances in the area of Lévy processes and their applications. Its goal is the dissemination of important developments in theory and applications. Each volume will describe state-of-the-art results of this rapidly evolving subject with special emphasis on the non-Brownian world. Leading experts will present new exciting fields, or surveys of recent developments, or focus on some of the most promising applications. Despite its special character, each article is written in an expository style, normally with an extensive bibliography at the end. In this way each article makes an invaluable comprehensive reference text. The intended audience are PhD and postdoctoral students, or researchers, who want to learn about recent advances in the theory of Lévy processes and to get an overview of new applications in different fields.

Now, with the field in full flourish and with future interest definitely increasing it seemed reasonable to start a series of Lecture Notes in this area, whose individual volumes will appear over time under the common name "Lévy Matters", in tune with the developments in the field. "Lévy Matters" appears as a subseries of the Springer Lecture Notes in Mathematics, thus ensuring wide dissemination of the scientific material. The mainly expository articles should reflect the broadness of the area of Lévy processes.

We take the possibility to acknowledge the very positive collaboration with the relevant Springer staff and the editors of the LN series and the (anonymous) referees of the articles.

We hope that the readers of "Lévy Matters" enjoy learning about the high potential of Lévy processes in theory and applications. Researchers with ideas for contributions to further volumes in the Lévy Matters series are invited to contact any of the editors with proposals or suggestions.

Aarhus, Denmark — Ole E. Barndorff-Nielsen
Paris, France — Jean Bertoin
Paris, France — Jean Jacod
Munich, Germany — Claudia Küppelberg
June 2010

A Short Biography of Paul Lévy

A volume of the series "Lévy Matters" would not be complete without a short sketch about the life and mathematical achievements of the mathematician whose name has been borrowed and used here. This is more a form of tribute to Paul Lévy, who not only invented what we call now Lévy processes, but also is in a sense the founder of the way we are now looking at stochastic processes, with emphasis on the path properties.

Paul Lévy was born in 1886, and lived until 1971. He studied at the Ecole Polytechnique in Paris and was soon appointed as professor of mathematics in the same institution, a position that he held from 1920 to 1959. He started his career as an analyst, with 20 published papers between 1905 (he was then 19 years old) and 1914, and he became interested in probability by chance, so to speak, when asked to give a series of lectures on this topic in 1919 in that same school: this was the starting point of an astounding series of contributions in this field, in parallel with a continuing activity in functional analysis.

Very briefly, one can mention that he is the mathematician who introduced characteristic functions in full generality, proving in particular the characterization theorem and the first "Lévy's theorem" about convergence. This naturally led him to study more deeply the convergence in law with its metric, and also to consider sums of independent variables, a hot topic at the time: Paul Lévy proved a form of the 0-1 law, as well as many other results, for series of independent variables. He also introduced stable and quasi-stable distributions, and unravelled their weak and/or strong domains of attractions, simultaneously with Feller.

Then we arrive at the book "Théorie de l'addition des variables aléatoires", published in 1937, and in which he summaries his findings about what he called "additive processes" (the homogeneous additive processes are now called Lévy processes, but he did not restrict his attention to the homogeneous case). This book contains a host of new ideas and new concepts: the decomposition into the sum of jumps at fixed times and the rest of the process; the Poissonian structure of the jumps for an additive process without fixed times of discontinuities; the "compensation" of those jumps so that one is able to sum up all of them; the fact that the remaining continuous part is Gaussian. As a consequence, he implicitly gave the formula

providing the form of all additive processes without fixed discontinuities, now called the Lévy-Itô Formula, and he proved the Lévy-Khintchine formula for the characteristic functions of all infinitely divisible distributions. But, as fundamental as all those results are, this book contains more: new methods, like martingales which, although not given a name, are used in a fundamental way; and also a new way of looking at processes, which is the "pathwise" way: he was certainly the first to understand the importance of looking at and describing the paths of a stochastic process, instead of considering that everything is encapsulated into the distribution of the processes.

This is of course not the end of the story. Paul Lévy undertook a very deep analysis of Brownian motion, culminating in his book "Processus stochastiques et mouvement brownien" in 1948, completed by a second edition in 1965. This is a remarkable achievement, in the spirit of path properties, and again it contains so many deep results: the Lévy modulus of continuity, the Hausdorff dimension of the path, the multiple points, the Lévy characterization theorem. He introduced local time, proved the arc-sine law. He was also the first to consider genuine stochastic integrals, with the area formula. In this topic again, his ideas have been the origin of a huge amount of subsequent work, which is still going on. It also laid some of the basis for the fine study of Markov processes, like the local time again, or the new concept of instantaneous state. He also initiated the topic of multi-parameter stochastic processes, introducing in particular the multi-parameter Brownian motion.

As should be quite clear, the account given here does not describe the whole of Paul Lévy's mathematical achievements, and one can consult for many more details the first paper (by Michel Loève) published in the first issue of the Annals of Probability (1973). It also does not account for the humanity and gentleness of the person Paul Lévy. But I would like to end this short exposition of Paul Lévy's work by hoping that this series will contribute to fulfilling the program, which he initiated.

Paris, France Jean Jacod

Preface

Statistics for stochastic processes is a topic in full development, driven by the needs of various applied fields, such as finance, bioscience or telecommunication. This volume of the series "Lévy Matters" is completely dedicated to this topic.

From an historical perspective, the topic started with the situation where the process under consideration is completely observed over some time interval $[0, T]$, and in the asymptotic theory the time horizon T goes to infinity. However, except for point or marked point processes, for which the times and sizes of the jumps are quite often observed, complete observation of the path over some time interval is possible in very rare cases only. Under almost all practical circumstances, a process can only be observed at discrete times, often equally spaced, or sometimes irregularly spaced. Recent mathematical advances allow us to deal with this situation of a discretely observed stochastic process, at least when this process has a nice structure, such as being a semimartingale or even an Itô semimartingale, and when the sampling scheme is regular. Irregular sampling schemes have also been considered, but they pose new challenges, especially when the sampling times are endogenous, that is, depend on the process itself. The Itô semimartingale assumption may appear, and is a serious mathematical restriction, but most models used by practitioners are of this type, because they are solutions of a stochastic differential equation driven by a Lévy process, or by a Brownian motion and a Poisson random measure.

A comprehensive statistical analysis of discretely observed Itô semimartingales is still far from being complete. However, the simplest semimartingales are Lévy processes, so a first step is to understand as well as possible the situation, when the underlying process is a Lévy processes observed at the times $i\Delta_n$ for $i = 0, 1, \dots, n$, with a mesh size Δ_n, which can be a constant (we then speak of *low frequency* observations), or is small and eventually goes to 0 as $n \to \infty$ (the *high frequency* setting). This setting is simple enough to allow for the development of efficient statistical tools, which hopefully can be extended to more general semimartingales, and it also plays the role of a benchmark, since any statistical procedure which works for semimartingales should *a fortiori* work for Lévy processes.

These reasons motivate the editing and writing of this volume, whose aim is to provide a rather extensive account on the most recent developments in the field of statistics for discretely observed Lévy processes.

Let us now be more specific. A Lévy process has a rather simple structure, as its law is completely characterized by three ingredients: the variance σ^2 of the Gaussian part, the drift b, and the Lévy measure F which describes the structure of the jumps, so the statistical problems amount to getting some information on the triple (b, σ^2, F). (This is in deep contrast with the general semimartingale case, for which the characteristics are *a priori* random, thus inducing non-standard statistical problems, where the "parameters" to estimate may be random.) So the main question here is how to estimate, in one way or another, the parameters b and σ^2, and also a parameter which may describe the family of Lévy measures in the model, or the measure F itself in a non-parametric way.

The observed increments $X_{i\Delta_n} - X_{(i-1)\Delta_n}$ are indeed i.i.d. variables, whose law only depends on the triple (b, σ^2, F) and on the mesh Δ_n. So in the low frequency setting $\Delta_n = \Delta$ we theoretically are on the known ground of the observation of an i.i.d. sample of variables. However, even in this case the problem is not trivial since we are after (b, σ^2, F) which, although in one-to-one correspondence with the distribution function of X_Δ (or with its density when it exists), has almost never an explicit form given in terms of these. In the high-frequency case, the observed increments are i.i.d., but their laws depend on n and become degenerate as $n \to \infty$.

The variance σ^2 has the distinctive property that it can be consistently estimated, in principle with the rate $\sqrt{n}$, whatever the asymptotic behaviour of Δ_n. In contrast, consistently estimating b and F requires $T_n := n\Delta_n \to \infty$, and the rates depend on T_n rather than n itself (typically $\sqrt{T_n}$ for b, whereas for F the rates are more complex to describe and strongly depend on the assumed hypotheses on F such as being a parametric family or a non-parametric family of finite measures, or other types of assumptions).

The three chapters below consider the statistical problem under different viewpoints:

Chapter "Estimation and Calibration of Lévy Models via Fourier Methods": D. Belomestny and M. Reiß study the low frequency situation. The method is based on the empirical characteristic function; they show in particular that estimators based on the empirical characteristic function enjoy rate-optimality for the two parameters b and σ^2, and also the optimal (minimax) non-parametric rate for the Lévy measure, mostly (but not only) in the case when the Lévy measure is finite. They also study the estimation of the so-called Blumenthal-Getoor index, which is a number in $[0, 2]$ and measures the degree of concentration of F near the origin, and the estimation when the observed process is not a Lévy process *stricto sensu*, but a time-changed Lévy process, which of course allows for a much wider range of applications.

Chapter "Adaptive Estimation for Lévy Processes": F. Comte and V. Genon-Catalot consider the non-parametric estimation, mainly in the high-frequency case with a time horizon $n\Delta_n$ going to infinity. All three quantities b, σ^2 and F are studied, although the emphasis is rather on the Lévy measure, assuming that it has a density, and as a general rule they find non-parametric estimators with better rates

than in chapter "Estimation and Calibration of Lévy Models via Fourier Methods", when expressed in terms of $T_n = n\Delta_n$. This is plausible, because in this situation one clearly has a better handgrip on the real size of the jumps. A distinctive feature of this part is the construction of *adaptive* estimators, based on deconvolution or projection or kernel methods.

Chapter "Parametric Estimation of Lévy Processes": H. Masuda, in contrast with the other authors, considers a completely parametric situation, when all three components c, σ^2 and F depend on a—possibly multidimensional—parameter θ. The emphasis is on maximum likelihood estimation and the Local Asymptotic Normality (LAN) property with a careful analysis of the various rates (for the drift, the diffusion and/or the Lévy measure), at which this property holds in the high-frequency case. He also proposes a method based on the median of suitably chosen functions of the observed increments, proving that (as for most methods of moments) it is rate-efficient. A large number of concrete example are treated in detail, showing how an actual implementation is possible.

Overall, these three chapters cover the main aspects of the estimation of discretely observed Lévy processes, when the observation scheme is regular, from an up-to-date viewpoint. We hope that the reader will find here a solid background on which statistical procedures for more general stochastic processes can be developed.

Aarhus, Denmark — Ole E. Barndorff-Nielsen
Zurich, Switzerland — Jean Bertoin
Paris, France — Jean Jacod
Munich, Germany — Claudia Küppelberg
August 2014

Contents