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Blocks of Finite Groups and Their Invariants

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Introduction

The classification of the finite simple groups is considered as one of the greatest achievements in mathematics of the twentieth century. The result provides the most basic pieces every finite group is composed of, and thus leads to a better understanding of symmetries arising from nature. The extremely long proof of the classification brings together the work of many mathematicians from different fields. One of the main contributors was Richard Brauer who introduced several innovative notions which became research topics on their own.

One of Brauer's ideas was to distribute the indecomposable representations of a finite group into its blocks. These blocks are algebras defined over an algebraically closed field of prime characteristic p. This shifts many problems about finite groups to questions about their blocks which are "smaller" speaking of dimensions. As an example, block theory was essentially used in Glauberman's famous Z^*-Theorem which in turn is a major ingredient in the proof of the classification mentioned above.

The present work focuses on numerical invariants of blocks and how they are determined by means of local data. Thus, we usually consider a block B of an arbitrary finite group G. Then it is a challenging task to determine the number $k(B)$ of irreducible representations of G in B. This global invariant is strongly influenced by a piece of local information called the defect group D of B. Here, D is a p-subgroup of G which is uniquely determined up to conjugation (and thus isomorphism). This raises the following natural question which will be our main theme:

What can be said about $k(B)$ and other invariants if D is given?

Brauer himself conjectured that the inequality $k(B) \le |D|$ should be true (here $|D|$ is the order of D). This problem, now known as Brauer's $k(B)$-Conjecture, has been unproved for almost 60 years. In this work we will give a proof of this conjecture under different types of additional hypotheses. These hypotheses often take the embedding of D in G into account. Therefore, we make extensive use of the language of fusion systems—a notion originally invented by Puig under the name Frobenius categories. In many instances the combination of old methods by Brauer

and Olsson using decomposition numbers together with new accomplishments from the theory of fusion systems turns out to be very successful.

Another even stronger conjecture from block theory, proposed by Alperin in 1986, makes a precise statement about the number $l(B)$ of simple modules of B in terms of so-called weights. We are able to obtain a proof of Alperin's Weight Conjecture for several infinite families of defect groups. In fact, these are the first new results of that kind after Brauer [41], Dade [65] and Olsson [212] settled blocks with finite and tame representation type over 20 years ago. Similarly, we provide evidence for Robinson's Ordinary Weight Conjecture which predicts the numbers $k_i(B)$ of irreducible characters of a given height $i \geq 0$. Note that $k(B)$ is the sum over the $k_i(B)$ $(i = 0, 1, 2, \ldots)$.

In some favorable cases we answer a more subtle question: What are the possible Morita equivalence classes of a block with a given defect group? If this can be done, we get an example of Donovan's Conjecture which asserts that there are only finitely many of these Morita equivalence classes. Here again our work represents the first advance after Puig's work [221] about nilpotent blocks and Erdmann's results [80] for the tame cases—both from the eighties. The verification of Donovan's Conjecture relies on the classification of the finite simple groups and thus fits in a recent development started by An, Eaton, Kessar, Malle and others (e.g. [7, 152]). In summary, the present work develops several powerful methods in order to tackle long-standing open conjectures in modular representation theory. The tools are far from being complete, but we hope to give a significant contribution which inspires further research.

We now describe the content of the book in detail. Of course, the first part serves as an introduction to the fundamentals of block theory of finite groups. In particular, we state Brauer's three main theorems, and we give a modern account on the notion of subpairs and subsections via fusion systems. Afterwards we present many open conjectures which all play a role in the following parts. Part II comprehends more sophisticated methods. The first section starts by introducing the notion of basic sets and other features attached to quadratic forms. Afterwards, I present the following general bound on $k(B)$ in terms of Cartan invariants:

$$k(B) \leq \sum_{i=1}^{l(b_u)} c_{ii} - \sum_{i=1}^{l(b_u)-1} c_{i,i+1}.$$

Here (u, b_u) is a so-called major subsection and (c_{ij}) is the Cartan matrix of b_u (for a more general version see Theorem 4.2). This bound, proved in [114], together with a practicable algorithm for computing Cartan matrices amounts to the "Cartan method"—one of the main tools for the upcoming applications. We also discuss as special cases Cartan matrices of small dimensions where our results still apply to arbitrary blocks. As an example, we obtain the implication

$$l(b_u) \leq 2 \Longrightarrow k(B) \leq |D|$$

where (u, b_u) is again a major subsection for B. This result from [252] generalizes an old theorem by Olsson [216] for the case $u = 1$. For the prime $p = 2$ we also prove Brauer's $k(B)$-Conjecture under the weaker hypothesis $l(b_u) \le 3$. Now let $p > 2$, and let (u, b_u) be an arbitrary subsection such that $l(b_u) = 1$ and b_u has defect q. Using the structure of the fusion system $\mathcal{F}$ of B we prove

$$k_0(B) \le \frac{|\langle u\rangle| + p^s(r^2 - 1)}{|\langle u\rangle| r} p^q \le p^q$$

where $|\mathrm{Aut}_{\mathcal{F}}(\langle u\rangle)| = p^s r$ such that $p \nmid r$ and $s \ge 0$. Here, $k_0(B)$ can be replaced by $k(B)$ whenever (u, b_u) is major. Finally, we take the opportunity to recall a less-known inequality by Brauer using the inverse of the Cartan matrix.

As another topic from this part we state Alperin's Fusion Theorem and deduce important properties of essential subgroups by invoking the classification of strongly p-embedded subgroups. These results are new for $p > 2$ and appeared in [257] in case $p = 2$. Afterwards, we collect material from the literature about the representation theory of finite simple groups. Here we indicate how to replace the arbitrary finite group G by a quasisimple group under suitable circumstances. The second part closes with a survey about p-blocks of p-solvable groups where we update an old structure result by Külshammer [161].

The third part of the present work gives applications to specific defect groups and represents the main contribution to the field. Its content assembles many recent papers of the present author, and also includes new results which have not appeared elsewhere. The content of these articles is strongly connected and we will freely arrange the material in order to improve readability. The chapter starts with the determination of the block invariants for metacyclic defect groups in case $p = 2$. This was mostly done in my dissertation (based on the work by Brauer and Olsson). But as a new result, we add a proof of Donovan's Conjecture for the abelian metacyclic defect groups which illustrates the power of the classification of the finite simple groups. Even more, this leads to infinitely many new examples supporting Broué's Abelian Defect Group Conjecture. Many of the other new results are likewise centered around defect groups which share properties of metacyclic groups. For odd primes p it is essentially harder to obtain the precise block invariants for metacyclic defect groups. However, as a consequence of a new result by Watanabe, Alperin's Weight Conjecture holds for all non-abelian metacyclic defect groups. Moreover, we are able to verify Brauer's Height Zero Conjecture which boils down to the inequality $k_0(B) < k(B)$ for non-abelian defect groups. This extends former results by Gao [88, 89], Hendren [107], Yang [287] and Holloway–Koshitani–Kunugi [120].

An obvious generalization of a metacyclic group is a bicyclic group, i.e. a group which can be written in the form $P = \langle x\rangle\langle y\rangle$ for some $x, y \in P$. It turns out that only for $p = 2$ we get new p-groups. Using a paper by Janko [140], we classify all fusion systems on bicyclic 2-groups. This leads to an interesting new result which states that a finite group is 2-nilpotent (and thus solvable) provided it has a bicyclic Sylow 2-subgroup P such that the commutator subgroup P' is non-cyclic. With the

list of all possible fusion systems in hand, we establish Olsson's Conjecture (i.e. $k_0(B) \le |D : D'|$) for all blocks with bicyclic defect groups.

Another project started in my dissertation focuses on minimal non-abelian defect groups D. Here D is non-abelian, but every proper subgroup of D is abelian. Using Rédei's classification [242] of these groups, we are able to complete the determination of the block invariants at least in case $p = 2$. As a byproduct we also reveal another example of Donovan's Conjecture for an infinite family of 2-groups. The proof of this result relies on the classification of the finite simple groups. For arbitrary primes p we show that Olsson's Conjecture holds for all blocks with minimal non-abelian defect groups, except possibly the extraspecial defect group of order 27 and exponent 3. This is also related to a theorem about controlled blocks with defect groups of p-rank 2 achieved in a different chapter.

Concerning Alperin's Weight Conjecture and Robinson's Ordinary Weight Conjecture, we give further evidence for several classes of 2-groups which are direct or central products of cyclic groups and groups of maximal class. Speaking of representation type these defect groups might be described as "finite times tame". We emphasize that apart from a small case the classification of the finite simple groups is not needed at this point. For the sake of completeness, we carry out computations for small defect groups as far as possible. The main achievement here is a proof of Brauer's $k(B)$-Conjecture and Olsson's Conjecture for the 2-blocks of defect at most 5. The former conjecture also holds for the 3-blocks of defect at most 3.

In Table 1 we collect many cases where the block invariants are known. Here we use the following abbreviations for three classes of bicyclic 2-groups:

$$\begin{aligned}
DC(m,n) &\cong \langle v,x,a \mid v^{2^n} = x^2 = a^{2^m} = 1,\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx\rangle \\
&\cong D_{2^{n+1}} \rtimes C_{2^m}, \\
DC^*(m,n) &\cong \langle v,x,a \mid v^{2^n} = 1,\ a^{2^m} = x^2 = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx\rangle \\
&\cong D_{2^{n+1}}.C_{2^m} \cong Q_{2^{n+1}}.C_{2^m}, \\
QC(m,n) &\cong \langle v,x,a \mid v^{2^n} = a^{2^m} = 1,\ x^2 = v^{2^{n-1}},\ {}^x v = {}^a v = v^{-1},\ {}^a x = vx\rangle \\
&\cong Q_{2^{n+1}} \rtimes C_{2^m}.
\end{aligned}$$

Moreover, $I(B) \cong \mathrm{Out}_{\mathcal{F}}(D)$ denotes the inertial quotient of the block B with defect group D.

As it is often the case, the study of these special cases leads to new ideas and general insights. This can be clearly seen in Chap. 14 where we improve the famous Brauer-Feit bound on $k(B)$ for abelian defect groups. The proof makes use of a recent result by Halasi and Podoski [101] about coprime actions. As a consequence, we are able to verify the $k(B)$-Conjecture for abelian defect groups of rank at most 5 (resp. 3) in case $p = 2$ (resp. $p \in \{3,5\}$). In the same spirit we show that Brauer's Conjecture remains true for arbitrary abelian defect groups whenever the inertial index of the block does not exceed 255. This result depends on perfect

Table 1 Cases where the block invariants are known

p	D	$I(B)$	Classification used?	References
Arbitrary	Cyclic	Arbitrary	No	Theorem 8.6
Arbitrary	Metacyclic, minimal non-abelian	Arbitrary	No	Theorem 8.13
Arbitrary	Abelian	$e(B) \leq 4$	No	[226, 227, 270]
Arbitrary	Abelian	S_3	No	[271]
≥ 7	Abelian	$C_4 \times C_2$	No	[273]
$\notin \{2, 7\}$	Abelian	C_3^2	No	[272]
2	Metacyclic	Arbitrary	No	Theorem 8.1
2	Maximal class $*$ cyclic, incl. $* = \times$	Arbitrary	Only for $D \cong C_2^3$	Theorems 9.7, 9.189.28, 9.37
2	Minimal non-abelian	Arbitrary	Only for one family where $\|D\| = 2^{2r+1}$	Theorem 12.4
2	Minimal non-metacyclic	Arbitrary	Only for $D \cong C_2^3$	Theorem 13.18
2	$DC(m, n)$ for $m, n \geq 2$	Arbitrary	No	Theorem 10.23
2	$DC^*(m, n)$ for	Arbitrary	No	Theorem 10.24
	$m, n \geq 2, m \neq n$			
2	$QC(m, n)$ for $m, n \geq 2$	Arbitrary	No	Theorem 10.25
2	$C_{2^n} \times C_2^3, n \geq 2$	Arbitrary	Yes	Theorem 13.9
2	$\|D\| \leq 16$	Arbitrary	Yes	Theorem 13.2
2	$C_4 \wr C_2$	Arbitrary	No	[160]
2	$D_8 * Q_8$	C_5	Yes	[252]
2	`SmallGroup`(32, 22)	Arbitrary	No	Proposition 13.10
2	`SmallGroup`(32, 28)	Arbitrary	No	Proposition 13.11
2	`SmallGroup`(32, 29)	Arbitrary	No	Proposition 13.11
3	C_3^2	$\notin \{C_8, Q_8\}$	No	[154, 282]

isometries constructed by Usami and Puig (e.g. [227, 270]) which reflect Broué's Abelian Defect Group Conjecture on the level of characters.

In the final chapter we address an inverse problem, i.e. we ask what can be said about defect groups D of B if the number $k(B)$ is given. Brauer's Problem 21 claims that there are only finitely many choices for D. An analysis of the situation $k(B) = 3$ leads to an interesting question about fusion systems with few conjugacy classes. We show that $k(B) = 3$ implies $|D| = 3$ provided the Alperin-McKay Conjecture holds. We also classify finite groups G such that all non-trivial p-elements in G are conjugate.

The present book has outgrown my habilitation thesis which was finished in 2013. I would like to thank Prof. Dr. Burkhard Külshammer for his constant support and encouragement. Further thanks go to Charles W. Eaton, Alexander Hulpke, Radha Kessar, Shigeo Koshitani, Jørn B. Olsson, Geoffrey Robinson, Ronald Solomon, and Robert Wilson for answering me specific questions. I am also grateful to Ines Spilling for her assistance in administrative tasks. Last but not least, I thank my mom for picking me up from the train station when I came back from California.

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