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Jacek Banasiak • Mustapha Mokhtar-Kharroubi
Editors

Evolutionary Equations with Applications in Natural Sciences

Springer

Editors
Jacek Banasiak
School of Mathematics, Statistics
and Computer Science
University of KwaZulu-Natal
Durban, South Africa

Mustapha Mokhtar-Kharroubi
Département de Mathématiques
UFR des Sciences et Techniques
Besançon, France

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Introduction

This book contains a collection of lectures presented at the 2013 CIMPA-UNESCO-South Africa School *Evolutionary Equations with Applications in Natural Sciences*. The School was part of the initiative *Mathematics of Planet Earth* and it was organized at the African Institute of Mathematical Sciences, http://www.aims.ac.za/, in Muizenberg (Cape Town), South Africa, from 22nd July to 2nd August 2013, under the auspices of the Centre International de Mathématiques Pures et Appliquées (CIMPA). CIMPA is a non-profit international organization whose aim is to promote international cooperation in higher education and research in mathematics for the benefit of developing countries. In accordance with CIMPA's mission, the School was aimed at postgraduate students and young researchers from such countries.

The School consisted of 9 courses delivered by invited lecturers coming from France, Germany, Poland, South Africa and Scotland, and attracted 29 participants from countries as diverse as Algeria, Benin, Cameroon, Chad, Congo, DRC, Ethiopia, Ivory Coast, Kenya, Lesotho, Mauritania, Morocco, Nigeria, Philippines, Pakistan, Sudan, South Africa, and Zimbabwe.

The School was a truly multidisciplinary event, spanning the fields of theoretical and applied functional analysis, partial differential equations, probability theory and numerical analysis applied to various models coming from theoretical physics, biology, engineering and complexity theory. The main emphasis was on the development of modelling, analytical and computational skills in a range of disciplines vital for the advancement of physical and natural sciences. The models discussed were time dependent and this led naturally to evolutionary equations which included ordinary, partial, integral or integro-differential equations. Particular examples of such equations were reaction-diffusion equations, transport equations coupled with Boltzmann type models and fragmentation-coagulation type equations, and the lectures gave detailed accounts of the functional-analytic, probabilistic and numerical frameworks for the analysis of such equations. This choice of programme was highly relevant to emerging researchers in many applied sciences as it exposed them to the diversity of applications of time dependent transport and kinetic type models and also provided a panorama of techniques for their analysis.

This volume contains the lectures given during the School, with each chapter devoted to the material presented by a specified lecturer. While each chapter is different and focuses on a specific topic, there is a common thread joining all of them—they are all concerned with evolution problems in a complex context and a significant part of each chapter deals with deep analytical methods for solving them. The lectures targeted postgraduate students and young researchers and thus the volume contains an appropriate blend of material, from an introductory and educational level at the beginning to a survey of cutting edge research at the end.

The foundation for all chapters is given by Wilson Lamb's *Applying functional analytic techniques to evolution equations*. This provides a gentle introduction to the basic tools needed for evolution equations and demonstrates their applicability. First the author discusses linear finite dimensional models, where the concept of the operator semigroup is introduced. Then he moves to nonlinear systems and discusses basic stability concepts. Finally, he discusses infinite dimensional models, both linear and nonlinear, illustrating abstract concepts with the discrete fragmentation-coagulation equation. We shall encounter the latter in a couple of other chapters, where the state-of-the-art theory of fragmentation and coagulation problems is developed.

The second chapter, *Boundary conditions in evolutionary equations in biology* by Adam Bobrowski, joins the narrative of the first at the concept of semigroups of operators, but takes this in a more general direction. First the author introduces basic models such as the transport and diffusion equations on the whole space and, writing them as Cauchy problems for abstract ordinary differential equations, discusses their solvability using the Hille–Yosida theory which is covered in more detail than in the first chapter. Next the author moves to the main subject of his lectures; that is, how to incorporate boundary conditions for the discussed equations so that they can be treated within the framework of semigroups of operators. Here he describes Greiner's approach and the method of images which are then applied to McKendrick models for population dynamics and to Feller–Wentzell boundary conditions. Finally, the author considers some singularly perturbed problems related to the previous models and discusses their small parameter limits using the powerful Sova–Kurtz approach.

The following chapter, *Introduction to complex networks: structure and dynamics* by Ernesto Estrada, takes us in the direction of discrete evolutionary problems. While, on the one hand, the author gives an introduction to the basic graph-theoretical concepts, including the 'small-world' and 'scale-free' networks, thus providing the mathematical foundations for the network transport problems considered in the next chapter, on the other hand he develops the theory of dynamical processes on networks, beginning with the consensus model, synchronization and Kuramoto models. The chapter is concluded by a discussion of a network version of the epidemiological SIR model introduced in the first chapter and the replicator-mutator model considered later in the lectures of R. Rudnicki.

In the next chapter, *Kinetic models in natural sciences* by Jacek Banasiak, we return to semigroup theory moving, however, to more complex models. In the study of so-called kinetic type equations (similar to Master Equations in Markov

processes) one considers the loss and gain of agents at a particular state and the solvability of such problems depends on a delicate balance of these terms. In this chapter the author considers transport processes on networks, building on the introduction by E. Estrada, discusses the nonlinear versions of the McKendrick model presented in the lectures of A. Bobrowski and also extends the theory of fragmentation-coagulation processes introduced in the lectures of W. Lamb. In the process, the author further develops semigroup theory, discussed in the first two chapters, by introducing concepts of positivity and analyticity of semigroups and employing these ideas in various ways to arrive at solutions of the problems formulated in the introductory part of the chapter. In particular, the author provides the first proof of the existence of global classical solutions to fragmentation-coagulation equation with unbounded rates.

Following these results, Philippe Laurençot in *Weak compactness techniques and coagulation equations* presents an alternative way of approaching coagulation problems. Whereas in previous chapters in which coagulation-fragmentation processes are discussed, the coagulation part is treated as a perturbation of the linear fragmentation part, thus enabling semigroup theory to be applied, here the centre stage is taken by the continuous coagulation operator. The author presents a powerful weak compactness technique, first discussing intricacies of weak L_1 convergence and state-of-the-art methods of dealing with them. Next he moves to the continuous coagulation equation, where he considers in detail the cases in which the weak solutions are mass conserving and the cases where a phase transition, called gelation, occurs. These results provide a powerful counterpoint to the semigroup approach described in the chapter by J. Banasiak, and it is particularly noteworthy that this book is the first in which both approaches are presented together.

The next two chapters, although similar in that they present theories concerning the long-term behaviour of solutions to abstract evolution equations are nevertheless quite diverse in their approach. The chapter *Stochastic operators and semigroups and their applications in physics and biology* by Ryszard Rudnicki focusses on methods that have their origins in probability theory. The author begins with the concept of stochastic operators and semigroups and introduces some examples such as the Frobenius–Perron operators and stochastic integral operators. For the former, he discusses their relations with the ergodic properties of the described system. In the next step, he discusses the concept of substochastic and stochastic semigroups, linking it nicely with the theory developed in the lectures of J. Banasiak. The examples that follow include birth-and-death type problems, the continuity equation, usual and degenerate diffusions (where the representation theorem by Hörmander is presented), as well as piecewise deterministic Markov processes (a special case of which is given by coupled systems of McKendrick models discussed in the lectures of A. Bobrowski and J. Banasiak). The final part of the chapter is devoted to the analysis of long-term properties of models fitting into the developed theory. Here the main roles are played by the Lasota–Yorke lower function method, partially integral semigroups, the Foguel alternative and the Hasminskiĭ function,

applied, among others, to cell cycle models, birth-and-death processes, and the description of the gene expression.

The chapter *Spectral theory for neutron transport* by Mustapha Mokhtar-Kharroubi focusses on a single model of neutron transport and is devoted to the classical approach to its analysis, spectral theory, which here is pushed to its limits. The author builds on the semigroup results developed in the lectures of W. Lamb, A. Bobrowski and J. Banasiak but leans more towards results on compactness of the semigroups and spectral mapping theorems which are further fine-tuned to cater for the advection and transport equations. The theory is first built for the advection equation where, when dealing with the natural L_1 space, the author makes a brief excursion into the so-called sun-dual theory. Then, to deal with the full equation, including the collision operator, he makes extensive use of perturbation theory and, in particular, Dyson–Phillips equations. Again, the L_1 theory is more difficult and is linked with the weak compactness results described in more detail in the lectures of P. Laurençot.

With the last two chapters, we move towards more concrete applications. The penultimate chapter *Reaction-diffusion-ODE models of pattern formation* by Anna Marciniak-Czochra begins with an introduction to pattern formation through Turing instabilities and discusses the example of an activator-inhibitor model, pointing out some limitations of the classical reaction-diffusion model. These limitations are addressed by introducing reaction-diffusion-ODE models for which an extensive theory of instabilities is developed. The theory presents a nontrivial extension of the semigroup methods for semilinear equations introduced in the lectures of W. Lamb and J. Banasiak, and of the spectral results discussed in detail in the preceding chapter by M. Mokhtar-Kharroubi. The author applies it to models such as early cancerogenesis and activator-inhibitor systems with non-diffusing activators. The final part of the chapter is devoted to pattern formation in systems with bistability and hysteresis and, in particular, to discontinuous patterns.

The final chapter, *Nonlinear Hyperbolic Systems of Conservation Laws and Related Applications* by Mapundi Banda, presents a vista into a numerical world. It begins with scalar conservation laws, the linear versions of which were discussed in the lectures of A. Bobrowski, J. Banasiak, R. Rudnicki and M. Mokhtar-Kharroubi, but quickly focuses on problems specific to quasi-linear cases such as weak, discontinuous solutions, formation of shock waves and entropy conditions. After introducing the basic toolbox for this field, the author moves to viscosity and entropy solutions for scalar conservation laws and then for systems. The main contents of the chapter are numerical methods for conservation laws. Here Godunov, Lax–Friedrichs and relaxation schemes are described in detail and applied to particular models. The chapter is concluded with numerical simulations of the flow in a network, thus providing an extension of some results discussed in the lectures of J. Banasiak to nonlinear models. The author also considers some aspects of the boundary stabilization method.

The Co-Directors of the School, Jacek Banasiak and Mustapha Mokhtar-Kharroubi, are deeply grateful to CIMPA for awarding the organization of the School to them, and to the Director of the African Institute of Mathematical Sciences

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Durban, South Africa — Jacek Banasiak
Besançon, France — Mustapha Mokhtar-Kharroubi

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