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Berkovich Spaces and Applications

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Introduction to the Volume: Berkovich Spaces and Applications

Antoine Ducros, Charles Favre, and Johannes Nicaise

1 Foreword

Most of the materials the reader will find in this book were presented during two scientific meetings which took place in January 2008 and June 2010 in Santiago de Chile and Paris, respectively. The Chilean workshop was organized by C. Favre, J. Kiwi and J. Rivera-Letelier, and focused on the interactions between dynamical systems and non-Archimedean geometry. The meeting in Paris was set up as a summer school to introduce young researchers from different backgrounds to the foundations of Berkovich's theory and its connections to number theory, model theory, Bruhat-Tits theory and dynamics. It was organized by A. Ducros, C. Favre, N. Fournaiseau, J. Nicaise and F. Paugam; the lectures were filmed, and the videos are available on the web site of the École normale supérieure, in the section *Savoirs en multimédia*.[1]

[1]URL : http://archives.diffusion.ens.fr/index.php?res=cycles&idcycle=490.

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Apart from some more specialized talks, the programs of these two meetings consisted of the following mini-courses:

– Santiago de Chile (January 2008):

  - ($*$) Jean-Pierre Otal: *Compactification of spaces of representations (after Culler, Morgan and Shalen)*;
  - Nicolas Ressayre: *Geometric invariant theory: constructing the moduli space of rational maps (after Silverman)*;
  - Antoine Ducros: *Introduction to Berkovich spaces*.

– Paris (June 2010):

  - ($*$) Michael Temkin: *Introduction to Berkovich analytic spaces*;
  - ($*$) Antoine Ducros: *Étale cohomology of schemes and analytic spaces*;
  - Vladimir Berkovich: $\mathbf{F}_1$-*geometry*;
  - François Loeser: *Model theory and analytic geometry*;
  - ($*$) Bertrand Rémy and Amaury Thuillier: *Bruhat-Tits buildings and analytic geometry*;
  - ($*$) Mattias Jonsson: *Dynamics on Berkovich spaces*;
  - Jérôme Poineau: *Berkovich spaces over* $\mathbf{Z}$.

The present volume contains expanded notes of the courses marked with ($*$), together with the contributions of A. Ducros on the cohomological finiteness of proper morphisms and C. Favre on sequential compactness of Berkovich spaces. It is meant as an introduction to Berkovich's theory and a selection of its applications to dynamics, geometry and number theory.

It is a great pleasure to thank all lecturers for their efforts to explain technically involved theories to a broad audience, and especially to those who made an extra effort to prepare their course notes for publication in these proceedings.

The two aforementioned conferences received financial support from many institutions including the Facultad de Matemáticas de la Pontificia Universidad Católica de Chile; the Institut de Mathématiques de Jussieu; the CNRS; the Universities Paris 6 Pierre et Marie Curie and Paris 7 Denis Diderot. They also benefited from the invaluable support of the projects ANR Berko in France, the Research Network on Low Dimensional Dynamics in Chile, and the joint French-Chilean ECOS/CONICYT project.

2 Introduction

2.1 Analytic Spaces Over a Non-Archimedean Field

Non-Archimedean geometry is the field of mathematics that aims to extend the methods and results from complex analytic geometry to non-Archimedean valued fields, with the field $\mathbf{Q}_p$ of p-adic numbers as a prime example, and to apply

these techniques to the study of various problems in geometry and number theory. There are different approaches to this aim, giving rise to different theories of non-Archimedean geometry. In this volume, the primary focus will lie on the theory developed by Berkovich at the end of the 1980s, whose foundations were established in [Ber90].

The most obvious way to develop a theory of analytic geometry over a non-Archimedean field K is to define analytic functions on open subsets of K^n as functions with values in K that can locally be written as converging power series. Once this class of functions is defined, one can use charts and atlases (or the language of locally ringed spaces) to define K-analytic varieties; see for instance [Gro] or Chap. 2 of [Igu]. More generally, one can include in such spaces points with coordinates in finite extensions of K; this leads to the notion of a *wobbly analytic space* in [Tat, §9].

These naive objects already have interesting applications in number theory, particularly through the use of p-adic integrals (see for instance [Igu]). However, the metric topology on a non-Archimedean field is fundamentally different from the metric topology on $\mathbf{C}$: the ultrametric property of the absolute value implies that the topology is totally disconnected. As a consequence, a purely local definition of an analytic function (like the one above) can never lead to a satisfactory global theory. For instance, every locally constant function on K is analytic, and there are many such functions that are not globally constant on K. This violates the principle of analytic continuation, which is one of the cornerstones of the complex analytic theory. Another problematic consequence of our naïve definition is that there are only very few isomorphism classes of K-analytic manifolds: if K is a non-Archimedean local field (i.e. a p-adic field or a finite extension of the field of Laurent series $\mathbb{F}_p((t))$), then Serre showed in [Ser] that every compact K-analytic manifold is isomorphic to a disjoint union of r open balls, with r a unique element in $\{0, \ldots, p-1\}$. Thus we cannot hope to recover a proper algebraic K-variety from its analytification, so that the GAGA principle fails in this setup. Even if we include non-K-rational points and consider wobbly spaces, similar problems arise: see for instance Corollary 2 in [Tat, §9].

To overcome these problems, J. Tate developed in the 1960s his theory of *rigid analytic spaces*. His main motivation was to provide a rigorous framework for his observations on the uniformization of p-adic elliptic curves with multiplicative reduction. The foundations of this theory were published in [Tat]. The underlying idea is to endow the wobbly spaces with some additional structure in order to *rigidify* them and restore the connection between local and global aspects. This is achieved by defining a Grothendieck topology on the wobbly space that only allows a particular type of covering. The theory of rigid analytic geometry, together with the necessary background from valuation theory and non-Archimedean analysis, is described in detail in [BGR]. Another introduction, which also discusses some interesting applications, can be found in [FvdP].

In the early 1970s, Raynaud showed in [Ray] how a certain class of rigid analytic spaces over K can be realized as generic fibers of formal schemes over the valuation ring R of K, and how different formal models of a fixed rigid

analytic space can be related via so-called admissible blow-ups (formal blow-ups whose centers are contained in the special fiber of the formal scheme). This result makes it possible to deduce many fundamental properties of rigid analytic spaces from the corresponding properties of formal schemes, for which an extensive algebro-geometric machinery is available. Raynaud's theory was systematically developed by Bosch, Lütkebohmert and Raynaud in the 1990s in the series of papers [BL93a,BL93b,BLR95a,BLR95b]. The foundations of the theory were recently laid out and expanded by Abbes in [Abb].

In 1990 the book [Ber90] of V. Berkovich appeared, where still another approach to non-Archimedean geometry was presented, based on spaces of absolute values with the topology of pointwise convergence. The distinguishing feature of these new analytic spaces is their particularly tame topology, which is both locally compact and locally arcwise connected. Moreover, the homotopy type of such an analytic space usually encodes very deep arithmetic-geometric information. With hindsight, one can say that it is precisely these properties that contributed to the success of this theory and its applicability in various (unexpected) fields including automorphic forms, compactification of buildings, singularity theory, complex dynamics and complex analysis.

In the introduction of [Ber90], Berkovich compares the construction of his spaces to the embedding of $\mathbf{Q}$ in $\mathbf{R}$. Pushing this analogy a bit further, we can compare rigid geometry to Berkovich's approach as follows. The metric topology on $\mathbf{Q}$ is totally disconnected, so that the algebra of continuous complex-valued functions on $\mathbf{Q}$ is rather pathological. One way to fix this problem is to define a Grothendieck topology on $\mathbf{Q}$ with the same open sets as the metric topology, but where the class of coverings of open subsets U of $\mathbf{Q}$ is restricted to coverings that can be refined to a locally finite covering by open intervals with rational endpoints (here locally finite means that every open interval with rational endpoints contained in U intersects only finitely many members of the covering). The archetypal example of a non-admissible open covering is the decomposition

$$\mathbf{Q} =]-\infty;\sqrt{2}[\,\cup\,]\sqrt{2};+\infty[.$$

It is not hard to see that this Grothendieck topology is connected. Furthermore the continuous complex-valued functions with respect to this Grothendieck topology (i.e., the naive continuous complex-valued functions such that the pull-back of *any* open covering of $\mathbf{C}$ is an admissible covering of $\mathbf{Q}$) are precisely the functions that extend to a continuous function on $\mathbf{R}$. The rigid approach corresponds to endowing $\mathbf{Q}$ with this Grothendieck topology, while Berkovich's approach corresponds to adding points to $\mathbf{Q}$ in order to obtain the topological space $\mathbf{R}$ itself, which has much better properties than $\mathbf{Q}$.

We should mention that there exist still other approaches to non-Archimedean geometry, each having its particular assets and applications, such as Huber's adic spaces [Hub] and the Riemann-Zariski spaces of Fujiwara and Kato [Kat]. Unfortunately, these theories fall outside the scope of this volume, but we strongly recommend the interested reader to consult the literature on these fascinating topics.

2.2 Literature

Beside Berkovich's original monographs [Ber90] and [Ber93], there are now several texts and surveys treating this subject at various levels of generality and difficulty. Let us mention [Ber98] and [Ber07] by Berkovich himself (the second one contains a very interesting account of his first steps in non-Archimedean geometry); the lecture notes [Con] and the survey [Nic], which both cover several approaches to non-Archimedean geometry (Tate, Raynaud, Berkovich); and the survey [Duc] about Berkovich's theory and its applications.

The text of M. Temkin in the present volume explains, in a systematic way and with some original insights, the foundations of the theory of Berkovich spaces and the relation with other theories of non-Archimedean geometry. It is more extensive and detailed than the surveys listed above.

2.3 Applications of Berkovich Theory

Since the origins of the theory, Berkovich spaces have been used in often surprising ways in several areas of geometry and number theory. Let us give a concise survey of some of these applications, with a view towards the chapters in this volume.

An important aspect of Berkovich geometry is the existence of a full-fledged theory of étale cohomology, which was published by Berkovich in [Ber93]. The first striking application of Berkovich theory to algebraic geometry appeared in 1994 when Berkovich proved a conjecture of Deligne concerning ℓ-adic vanishing cycles. Deligne's conjecture was deduced from a comparison theorem for étale vanishing cycles of formal schemes, which were defined by Berkovich using the theory of étale cohomology of analytic spaces. This theory has also become an important tool in the Langlands Program, where it is used to produce Galois representations over local fields in a geometric way.

The theory of étale cohomology of analytic spaces is carefully explained in the first text of A. Ducros, with a particular emphasis on the parallels (and some differences) between the étale cohomology of schemes and analytic spaces. No prior knowledge of the theory of étale cohomology of schemes is assumed.

Another remarkable feature of Berkovich geometry is that it works equally well over fields with the trivial absolute value. This makes it possible to use analytic tools in the study of algebraic schemes over arbitrary fields, by endowing the field with its trivial absolute value and applying the GAGA functor. As a toy example, one can obtain the étale cohomology spaces with compact support of an algebraic variety as derived functors of the functor of global sections with compact support, by first analytifying the algebraic variety (this is not true if one remains in the category of schemes). The second contribution of A. Ducros is a note that shows how to use Berkovich spaces over a field with the trivial absolute value to prove finiteness results in coherent cohomology for proper morphisms of noetherian

schemes. Ducros' strategy avoids the reduction to projective morphisms that is used in the classical approach.

A third class of applications of Berkovich spaces to algebraic geometry arise as follows. In many setups, one studies a geometric structure by constructing certain combinatorial invariants that reflect the geometry of the situation; the most obvious example is the theory of toric varieties and toroidal embeddings. It sometimes happens that these combinatorial invariants admit a natural interpretation in terms of valuations, so that they can be embedded in a suitable Berkovich space. One can then use Berkovich geometry to analyze the given geometric structure. An intriguing example is Thuillier's generalization of Stepanov's theorem in [Thu07]. Thuillier proves that the homotopy type of the dual complex of a log resolution of a pair of varieties (X, Y) over a perfect field k is independent of the log resolution. His strategy consists in showing that the dual complex Δ embeds into a certain analytic space over k with the trivial absolute value, attached intrinsically to (X, Y), and that Δ is a strong deformation retract of this Berkovich space.

Applications of a similar flavor appear in the theory of algebraic groups over local fields (Bruhat-Tits theory). Such applications were already explored by Berkovich in his foundational book [Ber90] in the split case. The survey by B. Rémy, A. Thuillier and A. Werner in this volume presents the main results obtained by the authors in a project where they extended Berkovich's first insights to an arbitrary algebraic reductive group G defined over a non-Archimedean local field. More precisely, they describe families of compactifications of the Bruhat-Tits building of G by observing that the building can be naturally embedded into the Berkovich analytification G^{an} of G.

A final field of applications of Berkovich spaces that we discuss is the theory of dynamical systems. Here one of the main ideas is to use Berkovich geometry to translate complex dynamical problems into the study of a dynamical system acting on a more tractable structure (for instance, an $\mathbb{R}$-tree) that lives inside a suitable Berkovich space.

This is explained in detail in the chapter by M. Jonsson, which discusses a series of (largely) unexpected applications of Berkovich theory to complex dynamical systems in low dimensions. Important technical tools in this work are the Laplace operator on $\mathbb{R}$-trees and the so-called subharmonic functions; their definitions are reviewed and put into a geometric perspective. This yields a gentle introduction to potential theory over non-Archimedean analytic spaces, which is treated more thoroughly in the book [BR] in the case of the Berkovich projective line, and by A. Thuillier in his Ph.D. thesis [Thu05] in the case of curves. We also refer to the series of recent papers [BFJ12a, BFJ12b, BPS, CL, CLD, Gub] for a glimpse of the recent struggles to develop such a potential theory over Berkovich analytic spaces of arbitrary dimension.

The text of J.-P. Otal explains how one can compactify the character variety of a group G (i.e., its space of representations into $\mathrm{SL}(2, \mathbb{C})$ modulo conjugacy) in a dynamically meaningful way. Points in the boundary correspond to representations

of G into $\mathrm{SL}(2,k)$ where k is a (non-Archimedean) valued extension of $\mathbb{C}$. This remarkable compactification was constructed by Morgan and Shalen in the 1980s, and the survey by Otal offers a fresh view on their work with applications to hyperbolic geometry in the vein of Thurston's ideas. Although Berkovich spaces are not explicitly mentioned in his text, a key role in this theory is played by $\mathbb{R}$-trees, which arise as Bruhat-Tits buildings of the group SL(2) and can be identified with subsets of the Berkovich projective line. Beside the obvious link with the article of Rémy, Thuillier and Werner, a whole section of Otal's paper is devoted to the notion of Riemann-Zariski space. This notion also plays an important role in the text by Temkin when studying the reduction of germs of analytic spaces, and in the short contribution by C. Favre on the sequential compactness of Berkovich spaces.

2.4 An Animated Introduction

We propose in this volume an introduction to Berkovich's theory that emphasizes its applications in various fields. A first part contains surveys of foundational nature (by M. Temkin, A. Ducros, C. Favre). A second part focuses on applications to geometry (A. Ducros, and B. Rémy, A. Thuillier and A. Werner). The third and final part explores the relationship between non-Archimedean geometry and dynamics (J.-P. Otal, M. Jonsson).

This is by no means a full account of the rapidly growing theory of Berkovich spaces, but we have tried to illustrate some of the main types of applications (as explained in the previous section). We hope that this book provides the reader with enough material on basic concepts and constructions related to Berkovich spaces to move on to more advanced research articles on the subject. We also hope that the applications presented here will inspire the reader to discover new settings where these beautiful and intricate objects might arise.

3 Plan of This Volume

We now provide a more detailed description of each of the articles in this book.

3.1 Part I: Berkovich Analytic Spaces

3.1.1 M. Temkin

Introduction to Berkovich Analytic Spaces

This text gives an overview of Berkovich's theory. The most technically involved proofs are omitted and replaced by precise references to the literature; many other

proofs are sketched, or formulated as exercises to the reader with some hints to a solution.

The text starts with the basic definitions of the theory (analytic spaces, morphisms, etc.), with some novelties: for instance, an alternative definition of analytic spaces is suggested (based upon the notion of a set-theoretic Grothendieck topology, instead of the language of atlases) and the notion of an H-strict space is introduced (it is a space defined using Zariski-closed subsets of polydiscs whose radii belong to a subgroup H of $\mathbf{R}_+^*$). Next, the author summarizes some basic facts about analytic spaces and their topology, and studies some important classes of morphisms (proper, smooth, flat, boundaryless, etc.).

Subsequently the author discusses the relations between Berkovich geometry and other theories. He explains the definitions and the basic properties of two natural functors (analytification of algebraic varieties, generic fibers of formal schemes). He compares Berkovich's viewpoint to other approaches to non-Archimedean geometry (Tate, Raynaud, Huber), and he shows how one can study analytic germs through Riemann-Zariski spaces (this method being due to the author).

The final chapter is devoted to analytic curves and explains, among other things, how the Semi-Stable Reduction Theorem helps to understand their homotopy types. The fundamental example of the affine line (which is investigated in full detail at the beginning of the paper) plays a crucial role here.

3.1.2 A. Ducros

Étale Cohomology of Schemes and Analytic Spaces

Étale cohomology was introduced in the scheme-theoretic context by Grothendieck in the 1950s and 1960s in order to provide a purely algebraic cohomology theory, satisfying the same fundamental properties as the singular cohomology of complex varieties, which was needed for proving the Weil conjectures. For deep arithmetic reasons related to the Langlands program, it turned out to be worthwhile to develop such a theory in the p-adic analytic context. This was done by Berkovich in the early 1990s.

After having given some general motivation, this article starts with the notion of a Grothendieck topology and its associated cohomology theory. Then it turns to the basic ideas and properties of étale cohomology for schemes and Berkovich spaces (these theories are closely related to each other), and to its fundamental results including various comparison theorems, Poincaré duality and so forth. Almost no proofs are given since most of them are highly technical and can be found in the references to the specialized literature provided in the text. Instead, the author has chosen to insist on examples, and shows how étale cohomology can be quite close to the classical topological intuition, while at the same time dealing in a completely natural manner with deep field-arithmetic phenomena (such as Galois theory). This often allows to think of arithmetic problems in a purely geometrical way.

3.1.3 C. Favre

Countability Properties of Berkovich Spaces

It is proven that compact Berkovich spaces are also sequentially compact when defined over the field of formal Laurent series in one variable. The proof is based on the extensive use of the Riemann-Zariski space of a variety, an object that plays a prominent role in the text of J.-P. Otal.

3.2 Part II: Applications to Geometry

3.2.1 A. Ducros

Cohomological Finiteness of Proper Morphisms in Algebraic Geometry: A Purely Transcendental Proof, Without Projective Tools

Originally, the coherent cohomological finiteness of proper morphisms between noetherian schemes was proven by reducing to the projective case, thanks Chow's lemma, and then using explicit computations, based upon the Čech complex associated with the standard affine covering of $\mathbb{P}^n$. In this short note, we give a new proof of this fact when the schemes are of finite type over a field k. The strategy consists in endowing the latter with the trivial valuation, then proving by hand a coherent GAGA-theorem in that setting (in a very concrete and explicit way), and eventually using the analytic coherent cohomological finiteness of proper morphisms, which was proven by Kiehl using non-Archimedean functional analysis.

3.2.2 B. Remy, A. Thuillier, and A. Werner

Bruhat-Tits Buildings and Analytic Geometry

Let G be a reductive algebraic group defined over a non-Archimedean local field k. During the 1960s and 1970s, F. Bruhat and J. Tits developed a new way to encode the properties of the group of rational points $G(k)$. They established a combinatorial description that can be stated in geometric terms using the *Euclidean building* of G over k. This Euclidean building is both a complete metric space and a simplicial complex and can be seen in many ways as a (singular) analogue of the Riemannian symmetric space of a semi-simple real Lie group. When Berkovich developed his theory of non-Archimedean geometry in the 1980s, he already mentioned the possibility of combining his theory with the theory of Bruhat-Tits. The crucial point is that the Bruhat-Tits building of G over k can be naturally embedded, as a topological space, into the Berkovich analytification G^{an} of G.

In this survey, the authors present the results of a joint project in which they develop and extend Berkovich's ideas. They show in particular how natural compactifications of the Bruhat-Tits building can be obtained through Berkovich geometry by procedures that are quite similar to Satake's theory for symmetric

spaces. The first part of the text reviews the necessary material from Bruhat-Tits theory, including basic definitions and properties of Euclidean buildings.

3.3 *Part III: Valuation Spaces and Dynamics*

3.3.1 M. Jonsson

Dynamics on Berkovich Spaces in Low Dimensions

The main goal of this survey is to review some dynamical properties of discrete dynamical systems acting on Berkovich analytic spaces of dimensions 1 and 2 and to describe some applications of this study to dynamical questions of algebraic nature.

- In the first part (Sects. 2–5), the author studies the dynamics of a rational map $R \in K(T)$ defined over an algebraically closed non-Archimedean valued field K. Since the work of Benedetto and Rivera-Letelier at the turn of the millennium, it is known that the natural setup for developing this theory is to look at the action of R on the Berkovich projective line $\mathbf{P}^1_{\mathrm{Berk}}$. The author focuses here on one special result of the theory: the fact that preimages of (most) points are asymptotically equidistributed with respect to a canonical R-invariant probability measure. This measure is supported in $\mathbf{P}^1_{\mathrm{Berk}}$ on the locus where the dynamics of R is unstable (the Julia set of R).

  Section 2 in the initial chapter is devoted to tree structures.[2] It also contains a construction of a potential theory on these objects, a tool that plays a fundamental role in the proof of the equidistribution result. Several equivalent definitions of $\mathbf{P}^1_{\mathrm{Berk}}$ are then given in Sect. 3, and its topology is carefully analyzed. The action of a rational map on the Berkovich projective line is described in Sect. 4. Basic results in local non-Archimedean dynamics, the construction of the dynamical partition of $\mathbf{P}^1_{\mathrm{Berk}}$ into the Fatou and the Julia set of R, the construction of the canonical measure, and the proof of the equidistribution theorem are finally given in Sect. 5.
- The second part (Sects. 6–10) contains a review of joint works of the author and C. Favre on the growth of two dynamical invariants associated with a polynomial map $f : \mathbf{A}^2 \to \mathbf{A}^2$. The first invariant $c(f)$ arises when f fixes a point, say the origin, and describes the local rate of contraction of f near this fixed point. The main result (Theorem B in Sect. 8) is that $c(f^n) \asymp c_\infty^n$ for some *quadratic integer* c_∞. The second invariant $\deg(f)$ is by definition the maximum of the degree of its components in some affine coordinates. Theorem C of Sect. 10 states that either $\deg(f^n) \asymp d_\infty^n$ for some quadratic integer d_∞ just as in the

[2]The intersection with Sect. 3 in the text of J.-P. Otal is nonempty but the main focus is quite different.

local case, or $\deg(f^n) \asymp n d_\infty^n$ with $d_\infty \in \mathbf{N}^*$ and f preserves a pencil of affine lines (up to conjugacy).

Both theorems are proved by looking at the dynamics of f on suitable subspaces of the Berkovich affine plane $\mathbf{A}^2_{\mathrm{Berk}}$ over a trivially valued field. For Theorem B, one considers the space $\mathcal{V}_0$ of semi-valuations centered at 0 and suitably normalized. This space is a tree, and its structure is described in Sect. 7. The key to the proof of Theorem B is the construction of a fixed point in $\mathcal{V}_0$. In the case of Theorem C, one looks at the set of semi-valuations in $\mathbf{A}^2_{\mathrm{Berk}}$ centered at ∞ and also suitably normalized. Its internal structure is tightly related to the geometry of the affine plane and is therefore more intricate than its local counterpart. It is described in detail in Sect. 9. The proof of Theorem C is given in Sect. 10.

Although this text does not contain any new results, it does present substantial simplifications of proofs as they were originally published. Interesting discussions in the text nicely complement the existing literature. A geometric interpretation of local degrees for rational maps in one variable is given in Sect. 4.6. The geometry of the Berkovich affine plane over a trivially valued field is described in an elementary way in Sect. 6. Arguments are given to extend results to the case of non-algebraically closed fields.

3.3.2 J.-P. Otal

Compactification of Spaces of Representations (After Culler, Morgan and Shalen)

These notes give an account of the work of Culler, Morgan and Shalen from the late 1980s concerning the compactification of the space of representations of a finitely generated group G to $\mathrm{SL}(2, \mathbf{C})$ modulo conjugacy, or, in other words, to the *character variety* $X(G)$ of this group. When G is the fundamental group of a 3-manifold M, then the geometry of $X(G)$ is intimately related to the existence of special geometric structures on M. For instance, the existence of a discrete fixed point free and faithful representation $G \to \mathrm{SL}(2, \mathbf{C})$ is equivalent to the existence of a hyperbolic metric on M.

The main goal of this survey is to explain how one can compactify $X(G)$ in a dynamically relevant way. Since $X(G)$ is an affine space over $\mathbf{C}$, the most naive idea would be to take its closure in a suitable projective space. This, however, does not lead to an interpretation of the boundary points that is suitable for applications. Culler, Morgan and Shalen designed a way to add points at infinity in $X(G)$ that correspond to representations of G into $\mathrm{SL}(2, F)$ for suitable non-Archimedean extensions F of $\mathbf{C}$. Any such representation then leads to an isometric action of the group G on the Berkovich projective line over F which turns out to be an $\mathbf{R}$-tree.

The presentation closely follows the original work of Morgan and Shalen and identifies the boundary of the compactification as (the image of) a suitable Riemann-Zariski space. The connection with Berkovich analytic spaces is not explicitly stated, but we refer to Sect. 5.7 of the paper of M. Temkin in this volume, or to

the contribution by C. Favre for a description of the link between Berkovich and Riemann-Zariski spaces.

As an application of these technics, J.-P. Otal gives a complete proof of a theorem due to Thurston that played a fundamental role in Thurston's approach to the geometrization of 3-manifolds. It states that the space of characters coming from discrete and faithful representations is relatively compact in $X(G)$ whenever G is the fundamental group of a compact boundary-incompressible and acylindrical 3-manifold.

This survey is essentially self-contained and recalls all facts from valuation theory and isometric actions on **R**-trees that are necessary for the proof of Thurston's theorem.

References

[Abb] A. Abbes, *Éléments de géométrie rigide, Volume I.* Construction et étude géométrique des espaces rigides. Progress in Mathematics, vol. 286 (Birkhäuser/Springer Basel AG, Basel, 2010), xvi+477 pp.

[BR] M. Baker, R. Rumely, *Potential Theory and Dynamics on the Berkovich Projective Line.* Mathematical Surveys and Monographs, vol. 159 (American Mathematical Society, Providence, RI, 2010), xxxiv+428 pp.

[Ber90] V.G. Berkovich, *Spectral Theory and Analytic Geometry Over Non-Archimedean Fields.* Mathematical Surveys and Monographs, vol. 33 (American Mathematical Society, Providence, RI, 1990), x+169 pp.

[Ber93] V.G. Berkovich, Étale cohomology for non-Archimedean analytic spaces. Publ. Math. IHES **78**, 5–161 (1993)

[Ber98] V.G. Berkovich, p-adic analytic apaces, in *Proceedings of the International Congress of Mathematicians*, Berlin, August 1998, Doc. Math. J. DMV, Extra Volume ICM II (1998), pp. 141–151

[Ber07] V.G. Berkovich, Non-Archimedean analytic geometry: first steps, in *p-adic Geometry: Lectures from the 2007 Arizona Winter School*, University Lecture Series, vol. 45 (American Mathematical Society, Providence, RI, 2008), pp. 1–7

[BGR] S. Bosch, U. Güntzer, R. Remmert, *Non-Archimedean Analysis. A Systematic Approach to Rigid Analytic Geometry.* Grundlehren der Mathematischen Wissenschaften, vol. 261 (Springer, Berlin, 1984), xii+436 pp.

[BL93a] S. Bosch, W. Lütkebohmert, Formal and rigid geometry. I. Rigid spaces. Math. Ann. **295**(2), 291–317 (1993)

[BL93b] S. Bosch, W. Lütkebohmert, Formal and rigid geometry. II. Flattening techniques. Math. Ann. **296**(3), 403–429 (1993)

[BLR95a] S. Bosch, W. Lütkebohmert, M. Raynaud, Formal and rigid geometry. III. The relative maximum principle. Math. Ann. **302**(1), 1–29 (1995)

[BLR95b] S. Bosch, W. Lütkebohmert, M. Raynaud, Formal and rigid geometry. IV. The reduced fibre theorem. Invent. Math. **119**(2), 361–398 (1995)

[BFJ12a] S. Boucksom, C. Favre, M. Jonsson, Solution to a non-Archimedean Monge-Ampère equation. Preprint: `arXiv:1201.0188`. To appear in J. Am. Math. Soc

[BFJ12b] S. Boucksom, C. Favre, M. Jonsson, Singular semipositive metrics in non-Archimedean geometry. Preprint: `arXiv:1201.0187`. To appear in J. Algebraic Geom

[BPS] J.I. Burgos Gil, P. Philippon, M. Sombra, Arithmetic geometry of toric varieties. Metrics, measures and heights. Astérisque **360**, vi+222 pages (2014)

[CL] A. Chambert-Loir, Mesure et équidistribution sur les espaces de Berkovich. J. Reine Angew. Math. **595**, 215–235 (2006)

[CLD] A. Chambert-Loir, A. Ducros, Formes différentielles réelles et courants sur les espaces de Berkovich. Preprint: `arXiv:1204.6277`

[Con] B. Conrad, Several approaches to non-Archimedean geometry. p-adic geometry, in *p-adic Geometry: Lectures from the 2007 Arizona Winter School*. Univ. Lecture Ser., vol. 45 (American Mathematical Society, Providence, RI, 2008), pp. 9–63

[Duc] A. Ducros, Espaces analytiques p-adiques au sens de Berkovich. Séminaire Bourbaki. Vol. 2005/2006. Astérisque No. 311 (2007), Exp. No. 958, viii, pp. 137–176

[FvdP] J. Fresnel, M. van der Put, *Rigid Analytic Geometry and Its Applications*. Progress in Mathematics, vol. 218 (Birkhäuser Boston, Boston, MA, 2004), xii+296 pp.

[Gro] A. Grothendieck, Techniques de construction en géométrie analytique, II: Généralités sur les espaces annelés et les espaces analytiques. Séminaire Henri Cartan, vol. 13 No. 1 (1960–1961), exp No. 9, 1–14

[Gub] W. Gubler, Local heights of subvarieties over non-Archimedean fields. J. Reine Angew. Math. **498**, 61–113 (1998)

[Hub] R. Huber, *Étale Cohomology of Rigid Analytic Varieties and Adic Spaces.* Aspects of Mathematics, vol. E30 (Friedr. Vieweg & Sohn, Braunschweig, 1996), x+450 pp.

[Igu] J. Igusa, *An Introduction to the Theory of Local Zeta Functions*. Studies in Advanced Mathematics, vol. 14, American Mathematical Society, Providence, RI; (International Press, Cambridge, MA, 2000), xii+232 pp.

[Kat] F. Kato, Topological rings in rigid geometry, in *Motivic Integration and Its Interactions with Model Theory and Non-Archimedean Geometry, I,* ed. by R. Cluckers, J. Nicaise, J. Sebag, London Mathematical Society Lecture Notes Series, vol. 383 (Cambridge University Press, Cambridge, 2011), xii+262 pp.

[Nic] J. Nicaise, Formal and rigid geometry: an intuitive introduction and some applications. Enseign. Math. (2) **54**(3–4), 213–249 (2008)

[Ray] M. Raynaud, Géométrie analytique rigide d'après Tate, Kiehl,. . .. Mémoires de la S.M.F. **39–40**, 319–327 (1974)

[Ser] J.-P. Serre, Classification des variétés analytiques p-adiques compactes. Topology **3**, 409–412 (1965)

[Tat] J. Tate, Rigid analytic spaces. Invent. Math. **12**, 257–289 (1971)

[Thu05] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*. Thesis, Université de Rennes 1, 2005. Available at `tel.archives-ouvertes.fr/docs/00/04/87/50/PDF/tel-00010990.pdf`

[Thu07] A. Thuillier, Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels. Manuscr. Math. **123**(4), 381–451 (2007)