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# Stabilization of Elastic Systems by Collocated Feedback

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# Introduction

In recent years an extensive literature was devoted to the controllability and stability of second order infinite dimensional systems coming from elasticity (see, for instance, Lions [91, 92], Komornik [73], Lasiecka-Triggiani [85], Lasiecka [81], Slemrod [120], Ammari-Tucsnak [23], Guo and his collaborators [62, 63], Komornik-Loretti [75], Coron [45], Tucsnak-Weiss [126], and references therein). According to the classical principle of Russell (see [118]) if a system is uniformly stabilizable forward and backward in time (plus some technical assumptions) by using collocated actuators and sensors, then it is exactly controllable by using the same actuators (i.e., the same input operator). We even refer to [113] for a very general formulation of this principle. The converse of this assertion is not true in general but it is true under specific conditions given in [47]. The only results available in the literature suppose that the input operator is bounded in the energy space (see Haraux [66]) or they are based on non local feed-backs (see, for instance, Komornik [74] and the references therein). Applied to PDE systems this situation leads to non-local feedbacks given in particular by Riccati-type operators. However for many PDE systems the exponential stability with collocated actuators and sensors was proved by direct methods using multiplier techniques (see Chen [39, 40], Lagnese [78], Komornik and Zuazua [76]).

The first aim of this book is to give a class of unbounded input operators for which exact controllability implies uniform stability by collocated actuators and sensors. The abstract setting (based on the results from [23]) is presented in Chap. 2 and its validation by concrete dissipative systems is given in Chap. 4.

Our mathematical framework is the following one. Let $X$ be a complex Hilbert space with norm and inner product denoted, respectively, by $||\cdot||_X$ and $(\cdot,\cdot)_X$. Let $A$ be a linear unbounded self-adjoint and strictly positive operator in $X$. Let $\mathcal{D}(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $(\mathcal{D}(A^{\frac{1}{2}}))'$ the dual space of $\mathcal{D}(A^{\frac{1}{2}})$ with respect to the pivot space $X$. Further, let $U$ be a complex Hilbert space (which is identified with its dual space) with norm and inner product, respectively, denoted by $||\cdot||_U$ and $(\cdot,\cdot)_U$ and let $B \in \mathcal{L}(U, (\mathcal{D}(A^{\frac{1}{2}}))')$.

Most of the linear control problems coming from elasticity can be written as

$$\begin{cases} x''(t) + Ax(t) + Bu(t) = 0, \\ x(0) = z^0,\ x'(0) = z^1, \end{cases}$$

where $(x, x') : [0, T] \to \mathcal{D}(A^{\frac{1}{2}}) \times X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function and for shortness the differentiation with respect to the time is denoted by "$'$".

We define the energy of the system at time $t$ by

$$E(t) = \frac{1}{2}\left\{||x'(t)||_X^2 + ||A^{\frac{1}{2}}x(t)||_X^2\right\}.$$

Simple formal calculations give

$$E(0) - E(t) = \int_0^t < Bu(s), x'(s) >_{\mathcal{D}(A^{\frac{1}{2}}),(\mathcal{D}(A^{\frac{1}{2}}))'} ds, \qquad \forall\ t \geq 0.$$

This is why, in many problems, coming in particular from elasticity, the input $u$ is given in the feedback form $u(t) = B^* x'(t)$, which obviously gives a non-increasing energy and which corresponds to collocated actuators and sensors.

Our aim is to give sufficient conditions making the corresponding closed-loop system

$$x''(t) + Ax(t) + BB^* x'(t) = 0, \tag{1}$$

$$x(0) = x^0,\ x'(0) = x^1, \tag{2}$$

uniformly stable in the energy space $\mathcal{D}(A^{\frac{1}{2}}) \times X$. In the case of non-uniform stability we give sufficient conditions for weaker decay properties.

In order to obtain the characterization of decay properties of the damped problem via observability inequalities for the conservative problem we will use a less restrictive assumption than the one in [66] which consists in the boundedness of the operator B. Our assumption corresponds to the boundedness of the transfer function

$$\lambda \to H(\lambda) = \lambda\, B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U),$$

on the line $\Re\lambda = \beta$, for some $\beta > 0$ (see below for the details). This approach goes back to [23,62]. Note that if the feedback operator $B$ is not suitable, the closed-loop system may even grow exponentially, see [47] for the details.

Alternatively a resolvent strategy can be used to tackle the stability of our problem. More precisely, using a standard technique of reduction of order, namely by introducing the unknown $U := (x,\ x')^\top$, system (1)–(2) reduces to

$$U' = \mathcal{A}_d U, \quad U(0) = U_0 = (x^0,\ x^1),$$

where the operator $\mathcal{A}_d$ is defined by

$$\mathcal{A}_d \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} u \\ -Ax - BB^* u \end{pmatrix}$$

with domain $\mathcal{D}(\mathcal{A}_d) = \left\{ (x,u)^T \in \mathcal{D}(A^{\frac{1}{2}}) \times X;\ Ax + BB^* u \in X \right\}$. It is well known (see, for instance, Theorem 1.1 of [31]) that the bounded $C_0$-semigroup $(e^{t\mathcal{A}_d})_{t \geq 0}$ on the Hilbert space $\mathcal{D}(A^{\frac{1}{2}}) \times X$ satisfies

$$\lim_{t \to +\infty} \|e^{t\mathcal{A}_d}(I - \mathcal{A}_d)^{-k}\| = 0, \tag{3}$$

for some/all $k \geq 1$ if and only if $\sigma(\mathcal{A}_d) \cap i\mathbb{R} = \emptyset$.

In [31], Duyckaerts and Batty prove a quantitative version of this result by indicating the rate of convergence in (3). They also unify various results scattered over the literature by first establishing an estimate of the convergence rate at infinity of the primitive of a bounded measurable function $f : \mathbb{R}_+ \to X$ towards the values of its Laplace transform $\hat{f}$ at 0. Indeed under the assumptions that there is a continuous and increasing function $M : \mathbb{R}_+ \to (0, \infty)$ such that its Laplace transform $\hat{f}$ extends analytically to the region

$$\{z \in \mathbb{C}\colon \operatorname{Re} z > -1/M(|\operatorname{Im} z|)\}$$

and satisfies the estimate $\|\hat{f}(z)\| \leq M(|\operatorname{Im} z|)$ throughout this region, then, as a special case of [31, Theorem 4.1], it follows that there exist positive constants $C$ and $T$ such that

$$\left\| \int_0^t f(s)\, ds - \hat{f}(0) \right\| \leq \frac{C}{M_{\log}^{-1}(t/C)} \quad \text{for every } t \geq T,$$

where

$$M_{\log}(s) = M(s)\,(\log(1 + M(s)) + \log(1 + s))$$

and $M_{\log}^{-1}$ is its inverse function.

This decay estimate applies to the function $e^{t\mathcal{A}_d}(I - \mathcal{A}_d)^{-k}$, see Theorem 1.1 of [31], and leads to sufficient conditions to get logarithmic, polynomial or exponential stability. Hence, Duyckaerts and Batty recover a result by N. Burq on logarithmic decay [36]. Moreover, they improve a result by Liu and Rao [97] and Bátkai et al. [29] on polynomial decay. Furthermore, their abstract result also allows to estimate the so-called truncated semigroups and in particular the local energy decay of solutions to the wave equations, as it is done by Burq [36].

In [31], the authors also conjecture that the decay estimates can be improved for semigroups on Hilbert spaces in the sense that a logarithmic factor can be dropped,

and that they are optimal in general Banach spaces. This conjecture is rigorously proved for the polynomial decay by Borichev and Tomilov in [35].

Another nice approach was recently described in the very interesting book of Jacob and Zwart [70]. It is based on a "port-Hamiltonian" formulation, which seems to be an uncommon concept within the mathematical community. The port-Hamiltonian systems are particularly well suited for the formulation of dynamical models of physical systems interacting with several other ones. For this class of systems, the general conditions for their well-posedness or stability, for instance, may be expressed in conditions that are easy to check, sometimes these conditions reduce to some simple conditions on matrices.

Time-delay often appears in many biological, electrical engineering systems and mechanical applications [5, 64, 121]. In many cases, in particular for distributed parameter systems, arbitrarily small delays in the feedback may destabilize the system, see, e.g., [30, 52–54, 65, 99, 107, 108, 115, 136]. Therefore, the stability issue of systems with delay is of theoretical and practical importance, the treatment of such issues can be found in [30, 77].

We further remark some similarities between techniques recently developed in [107, 108] in order to obtain some existence results and decay rates. We therefore propose to consider an abstract setting as large as possible in order to contain a quite large class of problems with time-delay feedbacks. In a second step we prove existence and stability results in this setting under realistic assumptions. Finally in order to show the usefulness of our approach, we give in Chap. 5 some examples where our abstract framework can be applied. This approach goes back to [109] but a similar one can be found in [12].

In the same Hilbert setting as before, for $i = 1, 2$, let $U_i$ be a real Hilbert space (which will be identified to its dual space) with norm and inner product denoted, respectively, by $\|.\|_{U_i}$ and $(.,.)_{U_i}$ and let $B_i \in \mathcal{L}(U_i, D(A^{\frac{1}{2}})')$.

We consider the system described by

$$\begin{cases} x''(t) + Ax(t) + B_1u_1(t) + B_2u_2(t-\tau) = 0, \ t > 0 \\ x(0) = z_0, \ x'(0) = z_1, \\ u_2(t-\tau) = f^0(t-\tau), \ 0 < t < \tau, \end{cases} \tag{4}$$

where $t \in [0, \infty)$ represents the time, $\tau$ is a positive constant which represents the delay, $(x, x') : [0, T] \to \mathcal{D}(A^{\frac{1}{2}}) \times X$ is the state of the system and $u_1 \in L^2([0, \infty), U_1), u_2 \in L^2([-\tau, \infty), U_2)$ are the input functions. Most of the linear equations modeling the vibrations of elastic structures with distributed control with delay can be written in the form (4), where $x$ stands for the displacement field.

In many problems, coming in particular from elasticity, the input $u_i$ are given in the feedback form $u_i(t) = B_i^*\dot{\omega}(t)$, which corresponds to collocated actuators and sensors. We obtain in this way the closed-loop system

$$\begin{cases} x''(t) + Ax(t) + B_1B_1^*x'(t) + B_2B_2^*x'(t-\tau) = 0, \ t > 0 \\ x(0) = x^0, \ x'(0) = x^1, \\ B_2^*x'(t-\tau) = f^0(t-\tau), \ 0 < t < \tau. \end{cases} \tag{5}$$

First we will give a sufficient condition that guarantees that this system (5) is well-posed, where we closely follow the approach developed in [107] for the wave equation. Secondly, we may ask if this system is dissipative. We show that the condition

$$\exists 0 < \alpha < 1,\ \forall u \in V,\ \|B_2^* u\|_{U_2}^2 \leq \alpha \, \|B_1^* u\|_{U_1}^2 \tag{6}$$

guarantees that the energy is decreasing; under this condition, using a result from [24] (see also [125]) we pertain a necessary and sufficient condition for the decay of the energy to zero. Note that this last condition is independent of the delay and therefore under condition (6), our system is strongly stable if and only if the same system without delay is strongly stable. Note further that if (6) is not satisfied, there exist cases where some instabilities may appear (see [107, 108, 136] for the wave equation). Hence this assumption seems to be realistic.

In a third step, again under the condition (6) and the boundedness of the transfer function (different from the one without delay)

$$\lambda \to H(\lambda) = \lambda \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} (\lambda^2 I + A)^{-1} (B_1,\ B_2) \in \mathcal{L}(U_1 \times U_2),$$

on the line $\Re\lambda = \beta$, for some $\beta > 0$, we prove that the exponential decay of the system (5) follows from a certain observability estimate. Again this observability estimate is independent of the delay term $B_2 B_2^* \dot{\omega}(t - \tau)$ and therefore, under these conditions, the exponential decay of the system (5) follows from the exponential decay of the same system without delay. A similar analysis for the polynomial decay is performed by weakening the observability estimate.

This book contains an introduction and five chapters. Chapter 1 contains some backgrounds and basic tools on functional analysis of linear semigroup theory, on diophantine approximations and about Ingham inequalities. In Chaps. 2 and 3, we consider abstract second order evolution equations with unbounded feedbacks with or without delay. Chapters 4 and 5 contain some applications to stabilization of concrete evolution systems with or without delay.

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