

Editors

J.-M. Morel, Cachan

B. Teissier, Paris

Advisory Board:

Camillo De Lellis (Zürich)

Mario di Bernardo (Bristol)

Alessio Figalli (Austin)

Davar Khoshnevisan (Salt Lake City)

Ioannis Kontoyiannis (Athens)

Gabor Lugosi (Barcelona)

Mark Podolskij (Aarhus)

Sylvia Serfaty (Paris and NY)

Catharina Stroppel (Bonn)

Anna Wienhard (Heidelberg)

More information about this series at
<http://www.springer.com/series/304>

Stefan Liebscher

Bifurcation without Parameters

Stefan Liebscher
Institut für Mathematik
Freie Universität Berlin
Berlin
Germany

ISBN 978-3-319-10776-9 ISBN 978-3-319-10777-6 (eBook)
DOI 10.1007/978-3-319-10777-6
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2014954087

Mathematics Subject Classification (2010): 34C23, 34C20, 34C37, 37G99, 35B32

© Springer International Publishing Switzerland 2015

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This monograph is devoted to the study of dynamical systems with manifolds of equilibria near points at which normal hyperbolicity of these manifolds is violated. It is targeted at mathematicians with at least basic familiarity with classical bifurcation theory. Although methods and concepts are briefly introduced, prior knowledge of center-manifold reductions and normal-form calculations may help to appreciate the presentation.

Manifolds of equilibria arise frequently in parameter dependent systems—by continuation of a trivial equilibrium. Loss of hyperbolicity of such equilibria yields qualitative changes of the local dynamics. Its study is one of the main objectives of classical bifurcation theory.

Here, however, we are interested in manifolds of equilibria that are not caused by additional parameters. Still, qualitative changes of the local dynamics close to the manifold of equilibria occur at points at which normal hyperbolicity of these manifolds breaks down. To exclude not only given but also any unknown or “hidden” parameters, we require the absence of any flow-invariant foliation transverse to the manifold of equilibria at the bifurcation point. We call the emerging theory *bifurcation without parameters*.

On first glance our setting appears to be very degenerate. Indeed, vector fields with manifolds of equilibria form a set of infinite codimension in the space of all smooth vector fields. However, there is a surprisingly rich and diverse collection of applications ranging from networks of coupled oscillators, viscous and inviscid profiles of stiff hyperbolic balance laws, standing waves in fluids, binary oscillations in numerical discretizations, population dynamics, memristor circuits, cosmological models, and many more.

Note that parameter dependent systems, likewise, form a set of infinite codimension in the space of vector fields with manifolds of equilibria—if we consider the parameters as fixed phase variables. As classical bifurcation theory is justified by its applicability, so is bifurcation theory without parameters.

This monograph is a slightly extended version of my Habilitation thesis at the Free University Berlin in 2013. I am much indebted to the Free University Berlin in

general and to Prof. Dr. Bernold Fiedler in particular for providing an environment where research and teaching, alike, is not only successful but also very enjoyable.

Acknowledgement

This work has been partially supported by the Collaborative Research Center 647 “Space—Time—Matter” of the German Research Foundation (DFG).

Berlin, Germany
May 2014

Stefan Liebscher

Contents

Part I Preliminaries

1	Introduction	3
1.1	Classical Bifurcation Versus Bifurcation Without Parameters	3
1.2	Manifolds of Equilibria	6
1.2.1	Conserved Quantities	6
1.2.2	Equivariances	7
1.2.3	Reversibilities	7
1.2.4	Singular Perturbations	8
1.2.5	Perturbing the Manifold	9
1.2.6	Cosymmetries	9
1.3	Classification of Bifurcation Types	10
1.3.1	Codimension One	10
1.3.2	Codimension Two	10
1.4	Further Cases	12
2	Methods and Concepts	13
2.1	Center Manifolds	13
2.2	Normal Forms	14
2.3	Normal Forms with Manifolds of Equilibria	16
2.4	Genericity	16
2.5	Unfoldings and Codimension	17
2.6	Rescaling and Blow Up	18
3	Cosymmetries	21

Part II Codimension One

4	Transcritical Bifurcation	27
4.1	The Generic Case	27
4.2	Additional Reflection Symmetry	30

5	Poincaré-Andronov-Hopf Bifurcation	35
6	Application: Decoupling in Networks	43
7	Application: Oscillatory Profiles in Systems of Hyperbolic Balance Laws	49

Part III Codimension Two

8	Degenerate Transcritical Bifurcation	57
8.1	Families of Lines of Equilibria: Singular Drift	57
8.2	Families of Lines of Equilibria: Fold	60
8.3	Planes of Equilibria	61
9	Degenerate Poincaré-Andronov-Hopf Bifurcation	67
9.1	Families of Lines of Equilibria: Singular Drift	68
9.2	Families of Lines of Equilibria: Fold	73
9.3	Planes of Equilibria	75
10	Bogdanov-Takens Bifurcation	81
10.1	Normal Form	82
10.2	Integrable Core	86
10.3	Poincaré Flow	87
10.4	Elliptic Integrals and the Ricatti Equation	88
10.5	Discussion of the Poincaré Flow	92
10.6	Poincaré Return Map and Bounded Solutions	96
11	Zero-Hopf Bifurcation	103
12	Double-Hopf Bifurcation	109
12.1	Family of Lines of Equilibria	109
12.2	Plane of Equilibria	112
13	Application: Cosmological Models of Bianchi Type, the Tumbling Universe	115
14	Application: Fluid Flow in a Planar Channel, Spatial Dynamics with Reversible Bogdanov-Takens Bifurcation	119
14.1	Fully Symmetric Case	122
14.2	Symmetry-Breaking Perturbations	124

Part IV Beyond Codimension Two

15	Codimension-One Manifolds of Equilibria	131
16	Summary and Outlook	135
16.1	Singularity Theory	135
16.2	Symmetries	136

16.3	Global Bifurcation.....	136
16.4	Recurrence	137
	References	139

List of Figures

Fig. 1.1	A normally hyperbolic line of equilibria with flow-invariant foliation	4
Fig. 4.1	Transcritical bifurcation	28
Fig. 4.2	Classical pitchfork bifurcation	31
Fig. 4.3	Transcritical bifurcation with reflection symmetry	32
Fig. 5.1	Poincaré-Andronov-Hopf bifurcation without parameters	37
Fig. 5.2	Poincaré-Andronov-Hopf bifurcation, splitting of separatrices	41
Fig. 6.1	Network of coupled oscillators	44
Fig. 6.2	Decoupling of a square ring of oscillators	47
Fig. 7.1	Oscillatory profile near elliptic Poincaré-Andronov-Hopf point	54
Fig. 8.1	Drift singularity along a one-parameter family of transcritical points	59
Fig. 8.2	Fold singularity along a one-parameter family of transcritical points	61
Fig. 8.3	Cusp singularity	64
Fig. 8.4	Transcritical point with drift singularity on a plane of equilibria...	65
Fig. 9.1	Andronov-Hopf point with drift singularity, subcritical case	71
Fig. 9.2	Andronov-Hopf point with drift singularity, supercritical case	72
Fig. 9.3	Fold of Andronov-Hopf points	75
Fig. 9.4	Degenerate Hopf point on a plane of equilibria	78
Fig. 10.1	Three cases of Bogdanov-Takens bifurcations without parameters	85
Fig. 10.2	Bogdanov-Takens point, integrable scaled flow, at order zero in ε	86

Fig. 10.3	Plots of the nonlinearities $2J_1 + 3\tilde{H}J_0, 3\tilde{H}J_1 + 4J_0,$ and $g(\tilde{H})$	88
Fig. 10.4	Bogdanov-Takens point, Poincaré flow	93
Fig. 10.5	Bogdanov-Takens point, splitting of manifolds	95
Fig. 10.6	Bogdanov-Takens point, Poincaré map, case (C), $\lambda > 0$	99
Fig. 10.7	Bogdanov-Takens point, Poincaré map, case (B), $\lambda > 0$	101
Fig. 11.1	Cases of Zero-Hopf bifurcation	107
Fig. 13.1	Kasner circle \mathcal{H} of equilibria and heteroclinic cap \mathcal{H}_1^+	117
Fig. 13.2	Kasner map.....	117
Fig. 14.1	Fluid flow in a plane channel.....	120
Fig. 14.2	Fully reversible Bogdanov-Takens point	123
Fig. 14.3	Reversible Bogdanov-Takens point, Poincaré flow	125
Fig. 14.4	Reversible Bogdanov-Takens point, Poincaré map, hyperbolic case.....	126
Fig. 14.5	Reversible Bogdanov-Takens point, heteroclinic orbits.....	126
Fig. 14.6	Reversible Bogdanov-Takens point, Poincaré map, elliptic case	127
Fig. 14.7	Set of bounded orbits in the Poincaré section, elliptic case.....	128